AN INTRODUCTION TO GEOMETRIC ALGEBRA AND GEOMETRIC CALCULUS

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Preface

An important tool for would-be mathematicians (and some other disciplines) is a knowledge of multivariable analysis. This has two major aspects, the analytical $\varepsilon - \delta$ side and the algebraic-geometric. There is a strong feeling in the mathematical community that the "right" way to master multivariable analysis ought to revolve about differential forms on manifolds. (This is on the algebraic-geometric side.)

Unfortunately, despite many excellent texts, such as [4], [13], and [25], the theory of differential forms has not succeeded in embedding itself in the consciousness of mathematicians and users of mathematics in the way that, say, linear algebra or differential equations has, nor in a way that would seem merited by its excellent properties and sophistication. (Exceptions to this can be found in subcommunities such as differential geometry or theoretical mathematical physics.)

This brings us to a second topic: geometric algebra. This concept grows from the work of H. Grassmann and W. K. Clifford (see [16, 5]) and is perhaps more properly thought of as being Clifford algebra in a particular setting. There is a vigorous, yet not widely visible, movement in the mathematical physics community to adopt geometric algebra and its attendant analytical machinery as the standard tool kit for analysis on manifolds. The great exponent of the movement over many years has been David Hestenes, and the book *Clifford Algebra to Geometric Calculus*, [22], by Hestenes and his colleague Garret Sobczyk, can appropriately be described as the "Bible" of the movement.

Hestenes is quite clear in his belief that geometric algebra (or, more precisely, geometric calculus or geometric function theory) is at least as powerful as differential form theory. Actually he says more than that; in [21], for example, we find, "I invite you, instead, to join me in proclaiming that Geometric Algebra is no less than a universal mathematical language for precisely expressing and reasoning with geometric concepts." The breadth and depth of his claims is impressive. A good summary of them can be found in [18, 20], while the evidence to support them is on display in [22].

Nor is his a lone voice in claiming a tremendous breadth, power, and potential importance for the machinery of geometric algebra and geometric calculus. Dorst et al., in *Geometric Algebra for Computer Science*, [9], say, "Geometric algebra is a powerful and practical framework for the representation and solution of geometrical problems. We believe it to be eminently suitable to those subfields of computer science in which such issues occur: computer graphics, robotics, and computer vision." From Doran and Lasenby, *Geometric Algebra for Physicists*, [8], with regard to mathematical tools for physics, we hear that "In this book we describe what we believe to be the most powerful available mathematical system developed to date." From Abłamowicz and Sobczyk's *Lectures on Clifford (Geometric) Algebras*, [1], we hear that, "Clifford (geometric) algebra offers a unified algebraic framework for the direct expression of the geometric ideas underlying the great mathematical theories of linear and multilinear algebra, projective and affine geometries, and differential geometry." And John Snygg, in *A New Approach to Differential Geometry Using Geometric Algebra*, [41], proclaims that, "The fact that Clifford algebra (otherwise known as "geometric algebra") is not deeply embedded in our current curriculum is an accident of history."

However writings on geometric algebra and geometric calculus at an introductory level are still in a process of being developed. Alan Macdonald has a particularly useful introduction and survey [33]. Macdonald has also brought out two books, [31, 32], that are suitable for undergraduates at the freshman, sophmore level. There is a text by Doran and Lasneby, [8], that is aimed more at the physics student than the mathematician or the general potential user of geometric algebra. The book [9] is targeted at computer scientists and provides a panoramic view of how to apply geometric algebra to various geometries. Garret Sobczyk's [42] approaches geometric algebra as an extension of the number concept. John Snygg, in [41], offers a serious attempt to use geometric algebra and geometric calculus as a tool in developing differential geometry. Two other offerings on the internet are [46] and the extensive notes [3].

There have been efforts to develop software for computation with geometric algebra and many examples can be turned up by simply running the phrase *geometric algebra software* on a search engine. Or a good list can be found at the website Geometric Algebra Software - Geometric Algebra Explorer with URL https://ga-explorer.netlify.com/index.php/ga-software/.

The purpose of these notes is to provide a limited exposure to the ideas of geometric algebra/calculus and to simultaneously use them as a vehicle to introduce the reader to some of the important concepts of multivariable analysis. (The emphasis here is on the geometric-algebraic side, and we do very little with $\varepsilon - \delta$ arguments.) Our target audience is mathematics students with the hope that they will also be accessible to a larger group, for example, physics, engineering, computer science students, etc. The required preparation is linear algebra, multivariable calculus from a decent introductory calculus course, and, of course, a modicum of mathematical sophistication.

Here are two interesting aspects of this enterprise:

This approach to multivariable analysis can be carried out in an unusually geometrical and physical way. That is, the machinery lends itself well to visualization and intuition, even in a higher-dimensional setting. In connection with that, an important genesis of these notes was the article [27] in *The American Mathematical Monthly* in which the idea of equivalence classes of oriented parallelepipeds is used to motivate and construct the space of k-vectors $\Lambda^k\mathbb{R}^n$ and the wedge product.

The second aspect is that the level of abstraction seems to be less high than is usually required for studying similar concepts. That may be partially because we have chosen to restrict ourselves to $\mathbb{R}^n$ equipped with the standard inner product of Euclidean space rather than dealing with more general Riemannian or pseudo-Riemannian manifolds. Let the reader judge for himself or herself the truth of these impressions.

Finally, I wish to acknowledge, on the one hand, the great personal encouragement and help I have received from Alan Macdonald in becoming acquainted with geometric algebra and geometric calculus and in preparing these notes and, on the other hand, the towering role of David Hestenes in making this material known to the world. I must also note the encouragement I have received from Stephen Kennedy as well as comments from certain anonymous reviewers which have led to improvements in the manuscript. If there are errors or shortcomings in what I have done, then the credit for those is wholly and solely mine.

Chapter 1

A little orientation

1.1 Things it would be well to know beforehand

Here is a somewhat more detailed list of the preparation one should have to read these notes.

Vector space axioms (over $\mathbb{R}$).
Linear independence, linear dependence, basis, dimension.
Linear transformation; representation as matrix.
Determinant; connection with independence.
$\mathbb{R}^n$ as a vector space.
Distance formula; magnitude of a vector.
Dot product in $\mathbb{R}^n$.
$a \bullet b = |a|\ |b|\ \cos\theta = \sum_{i=1}^{n} a_i b_i$.
Orthogonality.
Orthonormal basis.
Gram-Schmidt orthogonalization process.
Schwarz inequality.
Orthogonal transformation; rotation.
Lines, planes, $(n-1)$-dimensional analogues in $\mathbb{R}^n$; geometric significance of the equation

$$a \bullet (x - p) = 0 \quad \text{where } a, x, p \in \mathbb{R}^n.$$

Chain rule for partial derivatives of multivariable functions.
The chain rule as matrix multiplication.
Tangent vectors to curves.
The directional derivative.
The gradient and its significance in terms of rate of increase.
How to set up multivariate integrals as iterated integrals over simple regions.

1.2 Notations and conventions

We shall try to follow these conventions:

$\mathbb{R}$ is the real numbers.

$\mathbb{R}^n$ is the set of ordered n-tuples of real numbers.

$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ n times.

$\mathcal{I}$ is the unit interval $[0,1]$.

$\mathcal{I}^n$ is the n-dimensional unit cube $\mathcal{I} \times \cdots \times \mathcal{I}$.

Real numbers (scalars): Greek — α, β, etc.

Vectors, multivectors: Lower case Roman — a, b, x, etc.

Points, matrices: Upper case Roman — A, B, X, etc.

We shall take the *standard basis of* $\mathbb{R}^n$ to be $\{e_1, \ldots, e_n\}$ where

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0), \\ e_2 &= (0, 1, 0, \ldots, 0), \\ &\ldots \\ e_n &= (0, 0, \ldots, 0, 1). \end{aligned}$$

In $\mathbb{R}^3$ one often sees the standard basis vectors e_1, e_2, e_3 denoted $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively.

1.3 A word about presentation of proofs

Although we try to give a careful discussion of concepts and the logic behind them, there is a tendency in the first six chapters to move longer proofs to Appendix A. This is partly to keep things moving, partly because we wish to stress conceptual understanding over an examination of details.

Exceptions to this strategy occur when we spell out the construction of the geometric product and when we present the proof of the fundamental theorem of geometric calculus. Also, once we have gotten into Chapter 7, we tend to present the details of the proofs in all their messy glory.

1.4 Matrices and determinants

In all that follows, we restrict ourselves to vectors in $\mathbb{R}^n$ and make essential use of the dot product: Recall that if $a, b \in \mathbb{R}^n$ with $a = (\alpha_1, \ldots, \alpha_n)$ and $b = (\beta_1, \ldots, \beta_n)$, then $a \cdot b = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n$.

It will frequently be useful to talk about an $n \times k$ matrix $A = (a_1, \ldots, a_k)$ where each $a_i \in \mathbb{R}^n$. The matrix entries are not really specified without first giving a basis for the space. Thus, if we have in mind the basis $\{f_1, \ldots, f_n\}$ for $\mathbb{R}^n$ and we can write $a_i = \alpha_{1i} f_1 + \cdots + \alpha_{ni} f_n$ for each i, then we mean the matrix

$$A = (a_1, \ldots, a_k) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1k} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2k} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nk} \end{pmatrix}. \tag{1.1}$$

In (1.1) we have written A as $(a_1,\ldots,a_k)$ and we can think of each a_i as a column vector or column matrix,

$$a_i = \begin{pmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{ni} \end{pmatrix}$$

where the particular basis we are using, namely $\{f_i\}_{i=1}^n$, is in the back of our mind. Sometimes we want to write matrices in terms of row vectors. If, for example, we wrote the transpose of A, we might well put down

$$A^T = \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \ldots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \ldots & \alpha_{n2} \\ \vdots & \vdots & & \vdots \\ \alpha_{1k} & \alpha_{2k} & \ldots & \alpha_{nk} \end{pmatrix}.$$

It is often convenient to use a basis $\{f_1,\ldots,f_n\}$ for $\mathbb{R}^n$ having the property that $\{f_1,\ldots,f_k\}$ is a basis for span$\{a_1,\ldots,a_k\}$, in which case

$$A = (a_1,\ldots,a_k) = \begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1k} \\ \vdots & & \vdots \\ \alpha_{k1} & \ldots & \alpha_{kk} \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{pmatrix}.$$

We may then talk about the matrix

$$A' = \begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1k} \\ \vdots & & \vdots \\ \alpha_{k1} & \ldots & \alpha_{kk} \end{pmatrix}$$

associated with the vector subspace span$\{a_1,\ldots,a_k\}$, and, by an abuse of notation, we write $A' = (a_1,\ldots,a_k)$ even though A and A' are different size matrices.

We will usually write our matrices with respect to orthonormal bases because of an important connection between matrix multiplication and the dot product: If $A = (a_1,\ldots,a_k)$ and $B = (b_1,\ldots,b_m)$, where each a_i and b_j belongs to $\mathbb{R}^n$, and the matrices are specified with respect to an orthonormal basis for $\mathbb{R}^n$, then

$$B^T A = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} (a_1 \ldots a_k) = \begin{pmatrix} b_1 \bullet a_1 & \ldots & b_1 \bullet a_k \\ \vdots & & \vdots \\ b_m \bullet a_1 & \ldots & b_m \bullet a_k \end{pmatrix} \tag{1.2}$$

where B^T is, of course, the transpose of the $m \times k$ matrix B. A nice feature of this last matrix is that it is independent of the choice of basis. We also feel free to write it as $B^T A = (b_i \bullet a_j)$ or $(b_i \bullet a_j)_{m\times k}$.

We give exercises that establish (1.2) in detail.

The reader is also asked to establish in the exercises the useful fact that if either $\{a_1, \dots, a_k\}$ or $\{b_1, \dots, b_k\}$ is linearly dependent, then $\det(a_i \bullet b_j) = 0$. Because of this, later on in these notes, it sometimes makes sense to only deal with the case where $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ are sets of linearly independent vectors.

We usually treat the determinant in $\mathbb{R}^n$ as function of n vectors. Thus we write $\det(a_1, \dots, a_n)$ where we may imagine we are dealing with the $n \times n$ matrix $A = (a_1 \dots a_n)$ in which each a_i is a (column) vector. We recall from linear algebra the useful fact that the following properties *completely characterize* the determinant:

1. $\det(e_1, \dots, e_n) = 1$.
2. $\det(a_1, \dots, a_n)$ is linear in each variable a_i.
3. If we switch any two of the vectors a_i, then the determinant changes sign. For example,
$$\det(a_1, a_2, a_3, \dots, a_n) = -\det(a_2, a_1, a_3, \dots, a_n).$$
4. If $i \neq j$ and λ is a scalar, then
$$\det(a_1, \dots, a_{i-1}, a_i + \lambda a_j, a_{i+1}, \dots, a_n) = \det(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n).$$

(One can easily check these properties on 2×2 matrices.) It will also be convenient to keep in mind that $\det(A) = \det(A^T)$ for any square matrix A.

Exercises 1.4.

1. We let $a_1, a_2 \in \mathbb{R}^3$ be the vectors
$$\begin{aligned} a_1 &= 5e_1 + e_2 - 6e_3 \\ a_2 &= 3e_1 - 5e_3, \end{aligned}$$
and we construct a new basis for $\mathbb{R}^3$,
$$\begin{aligned} f_1 &= e_1 + e_2 - 2e_3 \\ f_2 &= e_1 - e_2 \\ f_3 &= e_3. \end{aligned}$$
Set
$$A = (a_1\, a_2) = \begin{pmatrix} 5 & 3 \\ 1 & -5 \\ -6 & 0 \end{pmatrix}$$
in terms of the standard basis. Now find $A' = (a_1\, a_2)$ in terms of f_1, f_2, f_3.
2. Suppose that $\{u_1, \dots, u_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

(a) Show that

$$\sum_{j=1}^{n}(e_i \bullet u_j)\,(e_k \bullet u_j) \;=\; \delta_{ik} \text{ (Kronecker's delta).}$$

(Hint: Expand e_i in terms of u_j and vice versa.)

(b) Suppose that $a, b \in \mathbb{R}^n$ and that we have expanded them in terms of both $\{e_i\}_{i=1}^n$ and $\{u_j\}_{j=1}^n$:

$$a \;=\; \sum_{i=1}^{n} \alpha_i\, e_i \;=\; \sum_{i=1}^{n} \gamma_i\, u_i,$$
$$b \;=\; \sum_{i=1}^{n} \beta_i\, e_i \;=\; \sum_{i=1}^{n} \delta_i\, u_i.$$

We know that $a \bullet b \stackrel{\text{def.}}{=} \sum_{i=1}^n \alpha_i\,\beta_i$. Show that

$$\sum_{i=1}^{n} \alpha_i\,\beta_i \;=\; \sum_{i=1}^{n} \gamma_i\,\delta_i.$$

(c) We choose two sets of vectors from $\mathbb{R}^n$: $a_1, \ldots, a_k$ and $b_1, \ldots, b_m$. Let us form the $n \times k$ and $n \times m$ matrices $A = (a_1 \cdots a_k)$ and $B = (b_1 \cdots b_m)$ where the entries for a_i and b_j come from their expansions in terms of e_k. Next let us form the matrices $A' = (a_1 \cdots a_k)$ and $B' = (b_1 \cdots b_m)$ where the entries now come from the expansions of the vectors in terms of u_i. Show that $B^T A = B'^T A'$.

3. Show that if $\{a_1, \ldots, a_k\}$ or $\{b_1, \ldots, b_k\}$ is linearly dependent, it follows that $\det(a_i \bullet b_j) = 0$.