# A Crash Course in Geometric Algebra and Geometric Calculus

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# Contents

# Introduction

These notes are meant to be a quick introduction to some of the ideas of geometric algebra and geometric calculus, somewhat in the spirit of Alan Macdonald's *A Survey of Geometric Algebra and Geometric Calculus*, [6]. Macdonald's paper can be obtained free at the indicated web address, it touches on topics which we do not cover, and it is highly recommended.

This is a *Crash Course* in that it is meant to be quick and superficial with essentially all proofs omitted. We hope that these notes will, nevertheless, give the reader *some* feel (however inadequate) for how geometric algebra and geometric calculus work.

The plan at this point is that this superficial crash course will be eventually followed by a much more detailed work [12].

In the meantime, the impatient reader who wants to get into the meat of the topic can consult the two introductory books [8] and [9] by Macdonald, the volume [3] by Hestenes and Sobczyk which is often considered the "bible" for this material, or a recent book [11] by Sobczyk. Those who would like to see an introductory book aimed at physicists may wish to look at [2] by Doran and Lasenby. There is also a set of online notes by Alan Bromborsky, [1], that seems aimed at a general mathematical audience.

The reader should have a modest acquaintance with linear algebra and a knowledge of multivariable calculus as presented in an introductory calculus course.

We shall use Greek letters, $\alpha$, $\beta$, $\chi$, $\eta$, and so forth for real numbers and Latin letters, $a$, $b$, $x$, $y$, $z$, etc. for vectors and multivectors. The symbol $\mathbb{R}$ stands for the set of real numbers and $\mathbb{R}^n$ for the set of ordered $n$-tuples of real numbers. Given $x = (\chi_1, \ldots, \chi_n)$ in $\mathbb{R}^n$, sometimes we think of this as a point, sometimes as a vector, depending on what we want to do.

Also, in $\mathbb{R}^n$, we shall use the symbols $e_1, \ldots, e_n$ for the *standard basis vectors*. That is, $e_i$ is the unit vector in the positive direction parallel to the $i$th-axis in $\mathbb{R}^n$. Thus in $\mathbb{R}^3$,

$$\begin{aligned} e_1 &= (1,0,0), \\ e_2 &= (0,1,0), \\ e_3 &= (0,0,1). \end{aligned}$$

# Chapter 1

# Simple $k$-Vectors

Our first step in understanding geometric algebra is to construct what are called $k$-vectors. In this chapter we look at special kinds of $k$-vectors, the *simple* $k$-vectors, which have a very nice geometrical interpretation. We move in the next chapter to general $k$-vectors.

Our presentation is rapid and superficial. Details about the construction of $k$-vectors and a number of proofs of their properties can be found in [5], *The Wedge Product and Analytic Geometry*, and in [12] (when it appears). A link to a copy of [5] can be found online at `http://www.mdeetaylor.com/?page_id=286`.

## 1.1 Definition

A vector $x = (\chi_1, \ldots, \chi_n)$ in $\mathbb{R}^n$ will also be referred to by us as a $1$-*vector*. It is usual to picture a vector (1-vector) as a *directed line segment*. This is useful for applications and helpful for the intuition.

Any two directed line segments are said to represent the same vector provided they have the same magnitude, direction, and orientation with respect to that direction. All such directed line segments may be considered to start at the origin; if you translate a directed line segment without altering its length, direction, or orientation, then it is considered to repesent the same vector. If we consider Figugre 1.1, then $a$ and $b$ represent the same vector. On the other hand, $c$ represents a different vector than $a$ because, although it has the same direction as $a$ in the sense that it is parallel to $a$, it is oriented in the opposite direction. And $d$ is not the same vector as $a$ because it

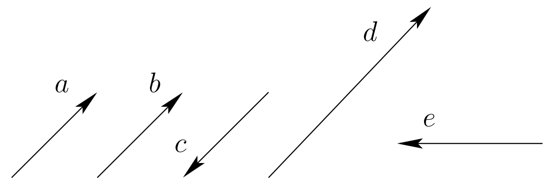


Figure 1.1: Vectors as directed line segements

has a different magnitude (length), while $e$ is different because it points in a different direction from $a$.

We want to introduce the idea of a $k$-*vector* where $k = 0, 1, 2, 3, \ldots$. Actually, what we shall discuss at this point is *simple* $k$-vectors.

The 1-vectors are just what we ordinarily think of as vectors. By 0-*vectors*, we mean scalars, that is, real numbers.

To see what we mean by a *simple* 2-*vector*, notice that if we choose two vectors $a_1$ and $a_2$ in $\mathbb{R}^n$, they determine a parallelogram. (Figure 1.2.) We

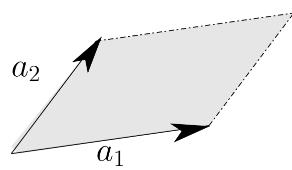


Figure 1.2: A simple 2-vector

think of this as being a representative of a simple 2-vector much in the sense in which we thought of a directed line segment as representing a vector. The symbol we use for this simple 2-vector is

$$a_1 \wedge a_2.$$

Just as with 1-vectors, we may think of simple 2-vectors as constructed from vectors starting at the origin. If we translate the associated parallelogram to a different location—translation in which we are careful not to rotate the parallelogram out of its defining plane—then it still represents the same simple 2-vector.

The three properties we associated with directed line segments have analogs here: magnitude, direction, orientation.

By the *magnitude* of $a_1 \wedge a_2$, we mean the area of the associated parallelogram. It turns out that the area of the parallelogram is given by

$$\text{magnitude of } a_1 \wedge a_2 \;=\; \sqrt{\det \begin{pmatrix} a_1 \bullet a_1 & a_2 \bullet a_1 \\ a_1 \bullet a_2 & a_2 \bullet a_2 \end{pmatrix}}$$

where $a_i \bullet a_j$ is the dot product of $a_i$ and $a_j$. (The knowledgeable reader may note that this looks suggestive of the distance formula in $n$-dimensional geometry. It should, and it generalizes.)

When we talk about two simple $2$-vectors—say, $a_1 \wedge a_2$ and $b_1 \wedge b_2$—then the analog to having the same direction is that they must both lie in the same plane, that is, in the same two-dimensional vector subspace $V$ of $\mathbb{R}^n$. (See Figure 1.3.) If they do not lie in the same two-dimensional vector subspace,

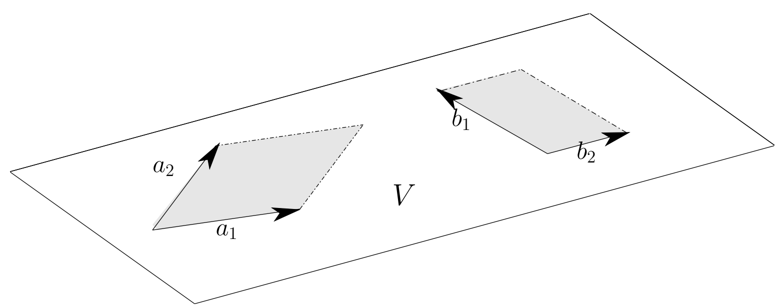


Figure 1.3: $a_1 \wedge a_2$ and $b_1 \wedge b_2$ lying in $V$

then they are considered to "have different directions."

As for orientation, this is analogous to both simple $2$-vectors being "right-handed" or "left-handed." The orientations of $a_1 \wedge a_2$ and $b_1 \wedge b_2$ can only be compared if they both lie in the same two-dimensional vector subspace. The reason for this is that if you pick $a_1 \wedge a_2$ up out of the plane in which it lives, flip it over in $3$-dimensional space, and drop it back into the plane, as in Figure 1.4, then it switches from being "right-handed" to "left-handed" and is now written as $a_2 \wedge a_1$.

If one has two simple $2$-vectors $a_1 \wedge a_2$ and $b_1 \wedge b_2$ lying in the same two-dimensional vector subspace $V$, then to see if they have the same orientation or not, one checks the sign of

$$\det \begin{pmatrix} a_1 \bullet b_1 & a_2 \bullet b_1 \\ a_1 \bullet b_2 & a_2 \bullet b_2 \end{pmatrix}. \tag{1.1}$$

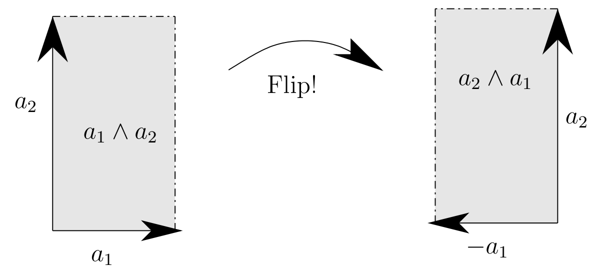


Figure 1.4: “Flipping” a 2-vector to change its orientation

If this is positive, then $a_1 \wedge a_2$ and $b_1 \wedge b_2$ are considered to have the *same orientation*; if it is negative, they have *opposite orientations*. If (1.1) is zero, then the parallelogram of one of the two simple 2-vectors (or perhaps both of them) must be degenerate, that is, a figure of zero area. If $a_1 \wedge a_2$ and $b_1 \wedge b_2$ lie in different planes, then we consider their orientations to be *incomparable*. Notice that the *order* in which $a_1$ and $a_2$ occur in the expression $a_1 \wedge a_2$ plays an important role here, and it is easy to check that unless the parallelogram is degenerate, $a_1 \wedge a_2$ and $a_2 \wedge a_1$ must always have opposite orientations.

Now here is the important point:

We consider $a_1 \wedge a_2$ and $b_1 \wedge b_2$ to be the *same simple* $2$*-vector* if and only if they have the same magnitude (area), direction (lie in a common plane), and orientation (handedness).

These considerations may be generalized to define *simple* $k$*-vectors*.

Suppose we have an ordered $k$-tuple $(a_1, \ldots, a_k)$ of vectors in $\mathbb{R}^n$; that is, each $a_i$ is a vector in $\mathbb{R}^n$ so can presumably be written in the form $a_i = (\lambda_{i1}, \ldots, \lambda_{in})$ where each $\lambda_{ij}$ is a real number. Each such ordered $k$-tuple generates a $k$-dimensional parallelepiped having $a_1, \ldots, a_k$ as its edges. (See Figure 1.2 for $k = 2$. For $k = 3$, see Figure 1.5.)

We leave it to the reader to play with the idea that each $x$ in $\mathbb{R}^n$ is a point of the parallelepiped if and only if we can write $x$ (considered as a vector) in the form $x = \tau_1 a_1 + \cdots + \tau_k a_k$ for some choice of scalars $\tau_1, \ldots, \tau_k$ where $0 \leq \tau_i \leq 1$ for each $i$. The *vertices* of the parallelepiped are those points where the scalars $\tau_i$ are all chosen to be $0$ or $1$.

Given a parallelepiped with edges $a_1, \ldots, a_k$, we take its $k$*-dimensional*

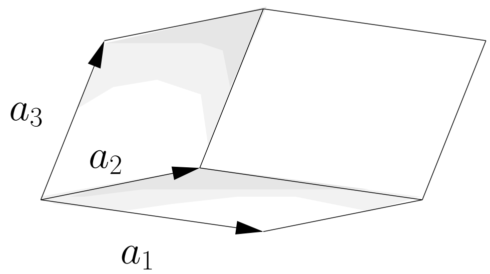


Figure 1.5: 3-dimensional parallelepiped

*volume* to be

$$\text{vol}(a_1, \ldots, a_k) \stackrel{\text{def.}}{=} \sqrt{\det \begin{pmatrix} a_1 \bullet a_1 & \cdots & a_1 \bullet a_k \\ \cdots & & \\ a_k \bullet a_1 & \cdots & a_k \bullet a_k \end{pmatrix}}.$$

This always turns out to be a nonnegative real number.

Of course,

$$1\text{-dimensional volume} = \text{length},$$
$$2\text{-dimensional volume} = \text{area}.$$

If vol$(a_1, \ldots, a_k) = 0$, then we say that the parallelepiped is *degenerate*. One can show that the parallelepiped with edges $a_1, \ldots, a_k$ is degenerate precisely when $a_1, \ldots, a_k$ are linearly dependent vectors. Thus the parallelogram with edges $a_1$ and $a_2$ is nondegenerate if and only if the parallelogram has positive area.

Now to generalize *orientation*. To do this, we want to think of a $k$-dimensional parallelepiped as being an *ordered* $k$-tuple of vectors $(a_1, \ldots, a_k)$. Each $a_i$ is an edge. The order in which they are written determines the orientation.

We do not directly define orientation; rather we define what it means to say that two $k$-parallelepipeds $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ to have the same orientation. This is sufficient for our purposes.

1. If $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are both linearly dependent sets, then $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are considered to have the same orientation, the *0-orientation*.

2. Suppose $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are both linearly independent sets and lie in the same $k$-dimensional vector subspace $V$ of $\mathbb{R}^n$. Then $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ have the *same (nonzero) orientation* provided
$$\det \begin{pmatrix} a_1 \bullet b_1 & \cdots & a_1 \bullet b_k \\ \vdots & & \\ a_k \bullet b_1 & \cdots & a_k \bullet b_k \end{pmatrix} > 0.$$
If $\det(a_i \bullet b_j)_{k\times k} < 0$, then they have *opposite orientations*.

3. In all other circumstances, the orientations of the two $k$-tuples are *non-comparable*.

We are now ready to define *simple $k$-vector*:

**Definition 1.** By the *simple $k$-vector* $a_1 \wedge \cdots \wedge a_k$, where $a_1, \ldots, a_k \in \mathbb{R}^n$ and $k \geq 1$, we mean the set of all ordered $k$-tuples $(b_1, \ldots, b_k)$ such that $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ have the same orientation and volume.

If $(a_1, \ldots, a_k)$ has the 0-orientation (or, equivalently, vol$(a_1, \ldots, a_k) = 0$), then we write $a_1 \wedge \cdots \wedge a_k = 0$.

We take $\mathbb{R}$ to be the set of simple $0$-vectors.

**Remark 1.** We feel free to write vol$(a_1 \wedge \cdots \wedge a_k)$ or vol$(a_1, \ldots, a_k)$ for the $k$-volume of a parallelepiped as the spirit moves us. It also turns out to be convenient to denote the volume thus,
$$|a_1 \wedge \cdots \wedge a_k| = \text{vol}(a_1 \wedge \cdots \wedge a_k),$$
when we think of $a_1 \wedge \cdots \wedge a_k$ as actually being a vector in an appropriate vector space and identify volume with the magnitude of that vector.

The following is very useful:

**Proposition 1.** *Let $a_1, \ldots, a_k$ be vectors in $\mathbb{R}^n$. Then the following are equivalent:*

*1.* $a_1 \wedge \cdots \wedge a_k = 0$.

2. *$a_1, \ldots, a_k$ are linearly dependent.*

3. *vol$(a_1 \wedge \cdots \wedge a_k) = |a_1 \wedge \cdots \wedge a_k| = 0$.*

*Similarly, these conditions are equivalent:*

1. *$a_1 \wedge \cdots \wedge a_k \neq 0$.*

2. *$a_1, \ldots, a_k$ are linearly independent.*

3. *vol$(a_1 \wedge \cdots \wedge a_k) = |a_1 \wedge \cdots \wedge a_k| > 0$.*

## 1.2 Operations with simple $k$-vectors

We define three operations with simple $k$-vectors: Multiplication by a scalar (a real number), the dot product, and the wedge product.

It should be noted that though we use the word "vector" in the expression "simple $k$-vector," these are not really vectors at all, at least not in the sense of belonging to a vector space. The problem is that we have no operation of *addition*. We shall remedy this lack in the next chapter.

**Definition 2.** For $\lambda \in \mathbb{R}$ and a $k$-blade $a_1 \wedge \cdots \wedge a_k$, we define $\lambda(a_1 \wedge \cdots \wedge a_k) = a_1 \wedge \cdots \wedge \lambda a_i \wedge \cdots \wedge a_k$ for $i = 1, \ldots, k$. By $-a_1 \wedge \cdots \wedge a_k$ we shall mean $(-1)(a_1 \wedge \cdots \wedge a_k)$.

In Figure 1.6 we show two different ways we can represent $2(a_1 \wedge a_2)$ as an oriented parallelogram. Of course the two oriented parallelograms are equivalent.

Scalar multiplication has an obvious geometric interpretation: If you multiply one edge of an oriented parallelepiped by $\lambda$, then the volume changes by a factor of $|\lambda|$. Thus

$$\begin{aligned}\text{vol}\big(\lambda(a_1 \wedge \cdots \wedge a_k)\big) &= \text{vol}(a_1 \wedge \cdots \wedge \lambda a_i \wedge \cdots \wedge a_k) \\ &= |\lambda|\, \text{vol}(a_1 \wedge \cdots \wedge a_k).\end{aligned}$$

If $\lambda > 0$, then the orientation is unchanged, but if $\lambda < 0$, then the orientation is reversed. So $a_1 \wedge \cdots \wedge a_k$ and $-a_1 \wedge \cdots \wedge a_k$ have opposite orientations but equal volume.

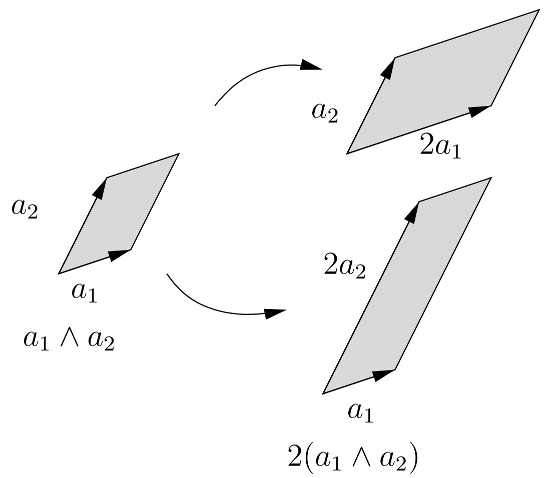


Figure 1.6: Two representations of multiplication by a scalar.

An important property of $a_1 \wedge \cdots \wedge a_k$ is that its volume is unchanged by a permutation of $a_1, \ldots, a_k$ however the orientation is changed by an *odd* permutation. Thus, for example,

$$\begin{aligned} a_1 \wedge a_2 &= -a_2 \wedge a_1, \\ a_1 \wedge a_2 \wedge a_3 &= -a_2 \wedge a_1 \wedge a_3 = a_2 \wedge a_3 \wedge a_1. \end{aligned}$$

A consequence of this is that if $a_i = a_j$ for distinct indices $i$ and $j$, then $a_1 \wedge \cdots \wedge a_k = 0$. The reason is that we must have

$$\begin{aligned} &a_1 \wedge \cdots \wedge a_i \wedge \cdots \wedge a_j \wedge \cdots \wedge a_k \\ &\qquad = -a_1 \wedge \cdots \wedge a_j \wedge \cdots \wedge a_i \wedge \cdots \wedge a_k \end{aligned}$$

since switching $a_i$ and $a_j$ requires an odd number of interchanges, but

$$\begin{aligned} &a_1 \wedge \cdots \wedge a_i \wedge \cdots \wedge a_j \wedge \cdots \wedge a_k \\ &\qquad = a_1 \wedge \cdots \wedge a_j \wedge \cdots \wedge a_i \wedge \cdots \wedge a_k \end{aligned}$$

since $a_i = a_j$.

Our next operation is the *dot product* of two simple $k$-vectors:

**Definition 3.**

$$(a_1 \wedge \cdots \wedge a_k) \bullet (b_k \wedge \cdots \wedge b_1) \overset{\text{def.}}{=} \det \begin{pmatrix} a_1 \bullet b_1 & \cdots & a_1 \bullet b_k \\ \vdots & & \\ a_k \bullet b_1 & \cdots & a_k \bullet b_k \end{pmatrix}.$$

Notice that on the left-hand side of the defining equation, we have written $b_k \wedge \cdots \wedge b_1$ rather than $b_1 \wedge \cdots \wedge b_k$. We are making use of *reversion* here. The reversion of a simple $k$-vector $b = b_1 \wedge \cdots \wedge b_k$ is

$$b^\dagger = (b_1 \wedge \cdots \wedge b_k)^\dagger \overset{\text{def.}}{=} b_k \wedge \cdots \wedge b_1.$$

Thus Definition 3 defines the dot product of

$$(a_1 \wedge \cdots \wedge a_k) \bullet (b_1 \wedge \cdots \wedge b_k)^\dagger.$$

The reason why this is convenient will be more easily seen once we introduce the *geometric product* later on.

**Example 1.** Recall that $e_1, e_2, e_3$ are the standard basis vectors of $\mathbb{R}^3$; that is, they are the unit vectors parallel to the $\chi_1$-, $\chi_2$-, and $\chi_3$-axes respectively oriented in the direction of increasing $\chi_i$. (You may have seen them before as $\mathbf{i}, \mathbf{j}, \mathbf{k}$.) Let

$$\begin{aligned} a_1 &= b_1 = e_1, \\ a_2 &= e_2 + e_3, \\ b_2 &= e_2. \end{aligned}$$

Then

$$\begin{aligned} (a_1 \wedge a_2) \bullet (b_2 \wedge b_1) &= \det \begin{pmatrix} a_1 \bullet b_1 & a_1 \bullet b_2 \\ a_2 \bullet b_1 & a_2 \bullet b_2 \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1. \end{aligned}$$

Here is a way in which the dot product of simple $k$-vectors resembles the dot product of vectors in $\mathbb{R}^n$ and in which it has a geometric interpretation: Suppose $a = a_1 \wedge \cdots \wedge a_k$ and $b = b_1 \wedge \cdots \wedge b_k$ simple $k$-vectors. It can be shown that

$$\left| a \bullet b \right| \leq |a|\,|b| = \mathsf{vol}(a_1 \wedge \cdots \wedge a_k)\,\mathsf{vol}(b_1 \wedge \cdots \wedge b_k).$$

Because of this, there is a unique $\theta$ such that $0 \leq \theta \leq \pi$ and

$$\cos(\theta) \;=\; \frac{a \bullet b}{|a|\,|b|}.$$

We define this $\theta$ to be the *angle between* the parallelepipeds that correspond to $a$ and $b$ or, more generally, to the vector subspaces of $\mathbb{R}^n$ that are determined by these parallelepipeds. (See Figure 1.7.)

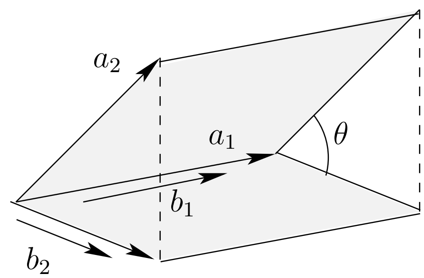


Figure 1.7: Angle between parallelepipeds

Our third operation is the *wedge product*:

**Definition 4.** Suppose we are given simple $k$- and $m$-vectors, $a = a_1 \wedge \cdots \wedge a_k$ and $b = b_1 \wedge \cdots \wedge b_m$ respectively. By their *wedge product* $a \wedge b$ we mean the simple $(k+m)$-vector

$$a \wedge b \;=\; a_1 \wedge \cdots \wedge a_k \wedge b_1 \wedge \cdots \wedge b_m.$$

Thus, for example,

$$(e_1 \wedge e_2) \wedge (e_1 \wedge e_3) \;=\; e_1 \wedge e_2 \wedge e_1 \wedge e_3 \;=\; 0$$

because $\{e_1, e_2, e_1, e_3\}$ is a linearly dependent set. On the other hand,

$$(e_1 \wedge e_3) \wedge e_2 \;=\; e_1 \wedge e_3 \wedge e_2 \;=\; -\,e_1 \wedge e_2 \wedge e_3$$

because a simple $k$-vector changes sign every time one interchanges two 1-vector factors.

# Chapter 2

# The Space of $k$-Vectors

## 2.1 The vector space $\Lambda^k\mathbb{R}^n$

We now permit ourselves to add simple $k$-vectors and call the results $k$-vectors. We write down *formal sums*; that is we put down expressions such as $3\,(e_1 \wedge e_2) + 1.5\,(e_1 \wedge e_3)$ and act as though we know what we are doing. Simple $k$-vectors have a geometric interpretation as equivalence classes of oriented parallelepipeds, but sums of simple $k$-vectors often lack such an interpretation. However one can do algebra with them, and this can make them very useful.

A careful justification for this operation of addition can be found in [5]. It is shown there that this can be done in such a way that we obtain a vector space which we designate $\Lambda^k\mathbb{R}^n$, the *space of $k$-vectors in* $\mathbb{R}^n$. That is, $\Lambda^k\mathbb{R}^n$ is vector space in that it satisfies all the axioms of a vector space: If we assume $a, b, c$ are $k$-vectors and $\lambda$ and $\xi$ are scalars, then

1. $a + (b + c) = (a + b) + c$.
2. $\lambda(a + b) = \lambda a + \lambda b$.
3. $(\lambda\xi)a = \lambda(\xi a)$.
4. $a + 0 = a$.
5. $a + b = b + a$.
6. $0\,a = 0$ and $1\,a = a$.

7. $a + (-a) = 0$.

Recall that $-a$ is $(-1)a$, and we may take $0$ to be the simple $k$-vector corresponding to a degenerate $k$-parallelepiped.

In the event that $k > n$, we take $\Lambda^k\mathbb{R}^n$ to be the vector space consisting of a single element, $0$. The reason for this is that if we are given a simple $k$-vector $a_1 \wedge \cdots \wedge a_k$ where $k > n$, then $\{a_1, \ldots, a_k\}$ *must* be a linearly dependent set, and thus $a_1 \wedge \cdots \wedge a_k = 0$.

We take $\Lambda^1\mathbb{R}^n$ to just be $\mathbb{R}^n$, and it is convenient to identify $\Lambda^0\mathbb{R}^n$ with the set of real numbers, $\mathbb{R}$.

## 2.2 Wedge product

We want to consider the special features of $\Lambda^k\mathbb{R}^n$.

The most outstanding of these is the existence of the *wedge product*: In the expression $a_1 \wedge \cdots \wedge a_k$, the symbol $\wedge$ is merely a part of our notation for a simple $k$-vector. However we do know that it is possible to form a wedge product of simple $k$- and $m$-vectors to form a simple $(k + m)$-vector:

$$(a_1 \wedge \cdots \wedge a_k) \wedge (b_1 \wedge \cdots \wedge b_m) \;=\; a_1 \wedge \cdots \wedge a_k \wedge b_1 \wedge \cdots \wedge b_m.$$

We can extend wedge as a binary operation to all $k$-vectors. Given $a, b, c$ which are $p$-, $q$-, and $r$-vectors respectively and $\lambda$ a scalar, we can construct the wedge product to have the following properties:

1. $a \wedge b$ is a $(p + q)$-vector.

2. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

3. $\lambda(a \wedge b) = (\lambda a) \wedge b = a \wedge (\lambda b)$.

4. Assuming $q = r$,
$$a \wedge (b + c) = a \wedge b + a \wedge c.$$
Assuming $p = q$,
$$(a + b) \wedge c = a \wedge c + b \wedge c.$$

It turns out to be convenient to treat scalars as $0$-vectors and to define the wedge product of a scalar $\lambda$ and a $k$-vector thus:

$$\lambda \wedge a \;=\; a \wedge \lambda \;=\; \lambda a.$$

**Example 2.** Using the properties listed above and the fact that interchanging the order of vectors in a simple $k$-vector changes the sign of the simple $k$-vector, we "simplify" a wedge product in $\mathbb{R}^4$:

$$\begin{aligned}-3e_2 \wedge &\big[(e_1 \wedge e_3) + 7(e_3 \wedge e_4)\big] \\ &= -3(e_2 \wedge e_1 \wedge e_3) - 21(e_2 \wedge e_3 \wedge e_4) \\ &= 3(e_1 \wedge e_2 \wedge e_3) - 21(e_2 \wedge e_3 \wedge e_4).\end{aligned}$$

## 2.3 Bases for $\Lambda^k \mathbb{R}^n$

The wedge product also has the following useful property: If $\{u_1, \ldots, u_n\}$ is a basis for $\mathbb{R}^n$, then the simple $k$-vectors of the form $u_{i_1} \wedge \cdots \wedge u_{i_k}$ where $i_1 < \cdots < i_k$ constitute a basis for $\Lambda^k \mathbb{R}^n$.

**Example 3.** We know that $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$, so bases for $\Lambda^k \mathbb{R}^3$ are as indicated:

$$\begin{aligned}\Lambda^2 \mathbb{R}^3 &: \ e_1 \wedge e_2, \quad e_1 \wedge e_3, \quad e_2 \wedge e_3. \\ \Lambda^3 \mathbb{R}^3 &: \ e_1 \wedge e_2 \wedge e_3.\end{aligned}$$

Notice that from the basis for $\Lambda^3 \mathbb{R}^3$, we see that dim$(\Lambda^3 \mathbb{R}^3) = 1$ and that if $a$ is any 3-vector in $\mathbb{R}^3$, it must have the form $a = \lambda(e_1 \wedge e_2 \wedge e_3)$ for some scalar $\lambda$.

**Example 4.** A basis for $\Lambda^2 \mathbb{R}^4$ is

$$e_1 \wedge e_2, \quad e_1 \wedge e_3, \quad e_1 \wedge e_4, \quad e_2 \wedge e_3, \quad e_2 \wedge e_4, \quad e_3 \wedge e_4.$$

Thus dim$(\Lambda^2 \mathbb{R}^4) = 6$.

This last example is a particular case of the following:

**Proposition 2.** *For $k = 0, 1, \ldots, n$, the dimension of $\Lambda^k \mathbb{R}^n$ is the binomial coefficient $\binom{n}{k}$.*

This is because, starting with a basis $\{u_i\}_{i=1}^n$ and forming basis $k$-vectors $u_{i_1} \wedge \cdots \wedge u_{i_k}$ where $i_1 < \cdots < i_k$, we see that we are concerned to count the number of sequences of length $k$, $(i_1, \ldots, i_k)$, that we can form by choosing from the set of $n$ objects $\{1, \ldots, n\}$.

This is perhaps a good place to introduce the idea of *multi-indices*. A multi-index is the same thing as an index except that it may have several entries. By a multi-index $I$ of length $k$, we mean a sequence $I = (i_1, \ldots, i_k)$ where each $i_r$ comes from some set of indices. Sometimes we drop the parentheses and write $I = i_1 \ldots i_k$. Thus, for example, if we consider a matrix with entries $\alpha_{ij}$, we can say that $ij$ is a multi-index of length $2$.

We will say that a multi-index $I = (i_1, \ldots, i_k)$ is *ordered* if $i_1 < \cdots < i_k$. Let $\{u_i\}_{i=1}^n$ be a basis for $\mathbb{R}^n$ and let us form a multi-index $I = (i_1, \ldots, i_k)$. then we set

$$u_I \stackrel{\text{def.}}{=} u_{i_1} \wedge \cdots \wedge u_{i_k}.$$

By our remarks about bases for $\Lambda^k \mathbb{R}^n$, we see that every $k$-vector $a$ has a unique expansion

$$a = \sum_I \lambda_I u_I$$

where the summation is over all the multi-indices of length $k$ that are ordered and the each $\lambda_I$ is a scalar.

## 2.4 Reversion and dot product

We can also apply the operation of reversion and take dot products of $k$-vectors.

Recall that the reversion of a simple $k$-vector is

$$(a_1 \wedge \cdots \wedge a_k)^\dagger = a_k \wedge \cdots \wedge a_1,$$

that is, we just reverse the order of the factors. We also know that $a_i \wedge a_j = -a_j \wedge a_i$. If we permute the factors of $a_1 \wedge \cdots \wedge a_k$ one at a time to obtain the reversion $a_k \wedge \cdots \wedge a_1$, then we find that

$$(a_1 \wedge \cdots \wedge a_k)^\dagger = (-1)^r a_1 \wedge \cdots \wedge a_k \quad \text{where} \quad r = \frac{k(k-1)}{2}.$$

We can define reversion for an *arbitrary* $k$-vector by extending the definition on simple $k$-vectors linearly. That is, we know that any $k$-vector $a$ can be expanded into a linear combination of simple $k$-vectors, $a = \sum_I \lambda_I a_I$ where each $a_I$ is simple. Then

$$a^\dagger \;=\; \sum_I \lambda_I a_I^\dagger.$$

An equivalent description is

$$a^\dagger \;=\; (-1)^r \sum_I \lambda_I a_I \quad \text{where} \quad r = \frac{k(k-1)}{2}.$$

In same way, we extend the definition of dot product linearly from simple $k$-vectors to arbitrary $k$-vectors.

That is, we know that for simple $k$-vectors, we have

$$(a_1 \wedge \cdots \wedge a_k) \bullet (b_1 \wedge \cdots \wedge b_k)^\dagger \;=\; \det \begin{pmatrix} a_1 \bullet b_1 & \cdots & a_1 \bullet b_k \\ \vdots & & \\ a_k \bullet b_k & \cdots & a_k \bullet b_k \end{pmatrix}.$$

If we have arbitrary $k$-vectors $a = \sum_I \lambda_I a_I$ and $b = \sum_J \xi_J b_J$ where $\lambda_I, \xi_J$ are scalars and $a_I, b_J$ are simple $k$-vectors, then we take the dot product of $a$ and $b$ to be

$$\begin{aligned} a \bullet b^\dagger \;&=\; \Big(\sum_I \lambda_I a_I\Big) \bullet \Big(\sum_J \xi_J b_J^\dagger\Big) \\ &= \sum_I \sum_J \lambda_I \xi_J (a_I \bullet b_J^\dagger). \end{aligned}$$

We now discover a nice game we can play with dot products if we have an orthonormal basis.

Let $\{u_i\}_{i=1}^n$ be an orthonormal basis for $\mathbb{R}^n$. We form the corresponding basis $k$-vectors $u_I$ for $\Lambda^k \mathbb{R}^n$ where $I$ ranges over all ordered multi-indices of length $k$. We find that if $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$, then

$$\begin{aligned} &\Big(u_{i_1} \wedge \cdots \wedge u_{i_k}\Big) \bullet \Big(u_{j_1} \wedge \cdots \wedge u_{j_k}\Big)^\dagger \\ &\qquad = \begin{cases} 1 & \text{if } (i_1, \ldots, i_k) = (j_1, \ldots, j_k), \\ 0 & \text{if } (i_1, \ldots, i_k) \neq (j_1, \ldots, j_k). \end{cases} \end{aligned}$$

That is, $\{u_I\}_I$, for suitable $I$, acts as an "orthonormal" basis for $\Lambda^k\mathbb{R}^n$. (The reader is invited to check this calculation for $k = 2$.) To put it slightly differently, if we use ordered multi-indices of length $k$, we have

$$u_I \bullet u_J = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

The following result is then a trivial calculation:

**Proposition 3.** *Suppose $\{u_i\}_{i=1}^n$ is an orthonormal basis for $\mathbb{R}^n$. Let $a, b \in \Lambda^k\mathbb{R}^n$ and let us expand them thus:*

$$a = \sum_I \lambda_I u_I, \quad b = \sum_J \xi_J u_J$$

*where $\lambda_I, \xi_J$ are scalars and the sums are over all ordered multi-indices $I, J$ of length $k$. Then*

1. $a \bullet a^\dagger = \sum_I \lambda_I^2$.
2. $a \bullet b^\dagger = \sum_I \lambda_I \xi_I$.

This proposition says that $\Lambda^k\mathbb{R}^n$ is isomorphic to $\mathbb{R}^m$ (where $m = \binom{n}{k}$) not only in terms of vector space properties but also with respect to the dot product. We take advantage of this "isomorphism" to define the *magnitude* of a $k$-vector $a$ by the equation

$$|a| \overset{\text{def.}}{=} \sqrt{a \bullet a^\dagger}.$$

We also find that the Cauchy-Schwarz inequality holds for $k$-vectors:

$$|a \bullet b| \leq |a|\,|b|.$$

Provided $a, b \neq 0$, we can use this to define an "angle" $\theta$ between $a$ and $b$ by

$$\cos(\theta) \overset{\text{def.}}{=} \frac{a \bullet b}{|a|\,|b|}$$

where $0 \leq \theta \leq \pi$. (We mentioned this previously for simple $k$-vectors.)

# Chapter 3

# Calculus With $k$-Vectors

We want to show some things one can do with $k$-vectors and calculus. However we start with a simple but useful application in analytic geometry.

## 3.1 Hyperplanes

A line $L$ in the plane or 3-dimensional space or more generally $\mathbb{R}^n$ is determined by two points. Let us call the points $a_0$ and $a_1$; we can also think of them as being vectors in $\mathbb{R}^n$. (Figure 3.1.) Notice that a point $x$ is on $L$

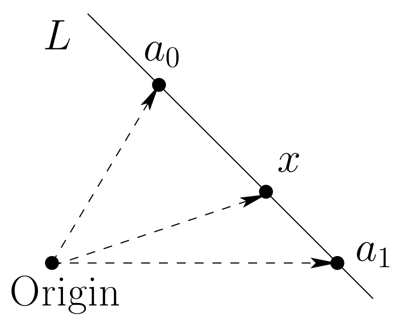


Figure 3.1: A line determined by two points

precisely when the vectors $a_1 - a_0$ and $x - a_0$ are linearly dependent, that is, when

$$(x - a_0) \wedge (a_1 - a_0) \ = \ 0.$$

In other words, this last equation can be considered an equation for the line. (It is a *nonparametric* equation; no parameters have been introduced to help us describe $L$.) Another way to specify $L$ is give a point that it passes through (for example, $a_0$) and a vector parallel to the line (for example, $b = a_1 - a_0$.) Then the equation for $L$ becomes

$$(x - a_0) \wedge b \ = \ 0. \tag{3.1}$$

**Example 5.** Let $L$ be the line in $\mathbb{R}^3$ passing through $a_0 = (1, 1, 2) = e_1 + e_2 + 2e_3$ in the direction given by the vector $b = 3e_1 - e_3$. The arbitrary point $x$ on $L$ is represented as $x = (\chi_1, \chi_2, \chi_3) = \chi_1 e_1 + \chi_2 e_2 + \chi_3 e_3$. Then Equation (3.1) becomes

$$\big[(\chi_1 - 1)e_1 + (\chi_2 - 1)e_2 + (\chi_3 - 2)e_3\big] \wedge (3e_1 - e_3) \ = \ 0.$$

Multiplying out, this reduces to

$$\begin{gathered} -3(\chi_2 - 1)\, e_1 \wedge e_2 + (-\chi_1 - 3\chi_2 + 7)\, e_1 \wedge e_3 \\ +(-\chi_2 + 1)\, e_2 \wedge e_3 \ = \ 0. \end{gathered}$$

Since $e_1 \wedge e_2$, $e_1 \wedge e_3$, and $e_2 \wedge e_3$ constitute a basis for $\Lambda^2\mathbb{R}^3$, the coefficients of the last equation must be zero, and Equation (3.1) ultimately reduces to the two scalar equations

$$\begin{aligned} \chi_1 + 3\chi_3 \ &= \ 7, \\ \chi_2 \ &= \ 1 \end{aligned}$$

which specify $L$.

Equation (3.1) can be generalized to describe any $k$-dimensional hyperplane in $\mathbb{R}^n$.

To say that $H$ is a $k$-dimensional hyperplane means that it must be a translate of a $k$-dimensional vector subspace of $\mathbb{R}^n$. To be more specific, let $V$ be a $k$-dimensional vector subspace of $\mathbb{R}^n$. Next choose a point $a_0$ in $\mathbb{R}^n$. If we want a hyperplane passing through $a_0$, we can *translate* $V$ to $a_0$. We do this by replacing every point $x$ of $V$ by $x + a_0$ and calling the resultant set $a_0 + V$; this is our hyperplane $H$.

Now since $V$ is a vector subspace and $k$-dimensional, we can find a basis $a_1, \ldots, a_k$ for $V$. These vectors will still be "parallel" to $a_0 + V$. Then an arbitrary point $x$ in $\mathbb{R}^n$ will belong to our hyperplane if and only if the vector $x - a_0$ is a linear combination of $a_1, \ldots, a_k$; that is, if and only if the vectors $x - a_0, a_1, \ldots, a_k$ are linearly dependent. This condition is captured by the equation

$$(x - a_0) \wedge (a_1 \wedge \cdots \wedge a_k) \;=\; 0. \tag{3.2}$$

## 3.2 Differentiability

We want to move beyond hyperplanes to more general surfaces. It is desirable to first indicate what we mean by *differentiability*.

If $\phi$ is a real-valued function with domain in $\mathbb{R}^n$, we say that $\phi$ is $\mathcal{C}^1$ at $x_0 = (\chi_{01}, \ldots, \chi_{0n})$ in the domain of $\phi$ provided each of the partial derivatives $\partial\phi/\partial\chi_i$ exists and is continuous in some open neighborhood of $x_0$. We say that $\phi$ is $\mathcal{C}^k$ at $x_0$ provided each of the mixed partials

$$\frac{\partial^k \phi}{\partial\chi_{i_1} \cdots \partial\chi_{i_k}}$$

exists and is continuous in some open neighborhood of $x_0$.

One of the most useful results of being $\mathcal{C}^k$ is that the order in which the differentiations are performed is irrelevant. Thus if $\phi$ is a function of $\chi$ and $\xi$ and is $\mathcal{C}^2$, we have

$$\frac{\partial^2\phi}{\partial\chi\,\partial\xi} \;=\; \frac{\partial^2\phi}{\partial\xi\,\partial\chi}.$$

We generalize the previous discussion in two ways: Suppose first of all that $f$ is a vector-valued function with domain in $\mathbb{R}^n$. We mean here that $f$ takes on values in $\mathbb{R}^n$ or some other $\mathbb{R}^m$; this generalizes the idea of a real-valued function. Second, instead of considering partial derivatives of $f$, we consider *directional derivatives*. We say that the *directional derivative of $f$ at $x_0$ in the direction specified by the vector $v$* is

$$\partial_v f(x_0) \;\stackrel{\text{def.}}{=}\; \lim_{\lambda\to 0}\frac{1}{\lambda}\big[f(x_0 + \lambda v) - f(x_0)\big] \tag{3.3}$$

where it is understood that $\lambda$ is a scalar. If $f$ is a function of the real variables $\chi_1, \ldots, \chi_n$, that is, we have $f(\chi_1, \ldots, \chi_n)$, then we get back to partial derivatives via the notation

$$\frac{\partial f}{\partial \chi_i} \stackrel{\text{def.}}{=} \partial_{e_i} f.$$

If $f : U \to \mathbb{R}^n$ where $U$ is an open subset of some $\mathbb{R}^m$, we say that $f$ is *differentiable at* $x_0 \in U$ if there is a linear transformation $f'(x_0) : \mathbb{R}^m \to \mathbb{R}^n$ and a function $g : V \to \mathbb{R}^n$ where $V$ is a neighborhood of $0$ in $\mathbb{R}^m$ such that

$$f(x_0 + v) - f(x_0) = \big[f'(x_0)\big]v + g(v) \quad \text{for } v \in V$$

and

$$\lim_{v \to 0} \frac{g(v)}{|v|} = 0.$$

$f$ is *differentiable on* $U$ if it is differentiable at every $x \in U$. We call the map $v \mapsto \big[f'(x_0)\big]v$ the *differential of* $f$.

When we look at this definition, we see that

$$\frac{1}{\lambda}\big[f(x_0 + \lambda v) - f(x_0)\big] \;=\; \big[f'(x_0)\big]v + \frac{g(\lambda v)}{\lambda}.$$

Thus if we let $\lambda \to 0$, we have

$$\big[f'(x_0)\big]v \;=\; \lim_{\lambda \to 0}\frac{1}{\lambda}\big[f(x_0 + \lambda v) - f(x_0)\big] \;=\; \partial_v f(x_0).$$

Here is a standard result from analysis:

**Proposition 4.** *If $f$ is $\mathcal{C}^1$ on the open set $U$, then it is differentiable at every point of $U$.*

## 3.3 Surfaces and tangent vectors

By a *surface*, we mean something like a surface of the type one encounters in introductory calculus or analytic geometry or, a little more generally a “manifold-with-corners." A hyperplane is an example of what we have in mind. We shall not give a precise definition of “surface” but shall say enough about the idea we have in mind that we can work with it.

The important thing about $p$-dimensional surfaces is that one can cover them with what we will call *coordinate patches* or *local parametrizations*, and these parametrizations are assumed to have certain nice properties.

If $x_0$ is a point in $\mathcal{M}$, we want to be able to find an open subset $U$ of $\mathbb{R}^p$ and a map $x : U \to \mathcal{M}$ such that

1. $x_0 \in x(U)$.
2. $x$ is one-to-one.

We say that $x$ or $x(U)$ (we are not very careful about the distinction) is a coordinate patch containing $x_0$. We also say that $x$ is a local parametrization of $\mathcal{M}$. Suppose the coordinates of a point $t \in \mathbb{R}^p$ are $(\tau_1, \ldots, \tau_p)$. If, under our coordinate patch $x$, we have $x_0 = x(t_0)$ and $t_0 = (\tau_{01}, \ldots, \tau_{0p})$, then we say that $x_0$ has the coordinates $(\tau_{01}, \ldots, \tau_{0p})$ with respect to this coordinate patch or parametrization.

If we want to say that $\mathcal{M}$ is $\mathcal{C}^r$ *surface* (where $r \geq 1$), we require that for each coordinate patch $x$, the maps $x$ and $x^{-1}$ be $\mathcal{C}^r$.

Now a little notation. Since we are letting $(\tau_1, \ldots, \tau_p)$ be the coordinates of $t \in \operatorname{dom}(x) \subseteq \mathbb{R}^p$, let us introduce the notation

$$\frac{\partial x}{\partial \tau_i} \stackrel{\text{def.}}{=} \partial_{e_i} x.$$

(We are, of course, assuming that $x$ is at least $\mathcal{C}^1$.) The way we have done this, $\partial x/\partial \tau_i$ should be a function of $t_0 \in \mathbb{R}^p$, but sometimes we shall abuse notation and write it as a function of $x_0 \in \mathcal{M}$. Since $x$ is a one-to-one map, we should not get into trouble.

Now here is an important point: Each $\partial x/\partial \tau_i$, when evaluated at $x_0$ (or $t_0$), is a *tangent vector* to $\mathcal{M}$ at the point $x_0$.

We may define a *tangent vector to* $\mathcal{M}$ thus:

$$\begin{aligned}&\text{tangent vector to } \mathcal{M} \text{ at } x_0 \\ &\qquad = \lambda_1 \left(\frac{\partial x}{\partial \tau_1}(x_0)\right) + \cdots \lambda_p \left(\frac{\partial x}{\partial \tau_p}(x_0)\right)\end{aligned} \tag{3.4}$$

where $\lambda_1, \ldots, \lambda_p$ are scalars. That is, it is a linear combination of the tangent vectors $\partial x/\partial \tau_i$ when evaluated at $x_0$. (Of course, this definition does not depend on which parametrization $x$ we use.) By $T_{x_0}\mathcal{M}$ we shall mean the space of all tangent vectors to $\mathcal{M}$ at $x_0$.

A second and sometimes useful way to construct a tangent vector to $\mathcal{M}$ at $x_0$ is this: Let $f : U \to \mathcal{M}$ be a map where $U$ is a subset of $\mathbb{R}$; in other words, $f$ defines a *curve* in $\mathcal{M}$. We assume $f$ passes through $x_0$; without loss of generality, we suppose that $f(0) = x_0$. Assuming further that $f$ is at least a $\mathcal{C}^1$ function, if

$$v \ = \ \lim_{\lambda\to 0} \frac{1}{\lambda}\big[f(\lambda) - f(0)\big] \ = \ f'(0),$$

then $v$ is a tangent vector to $\mathcal{M}$ at $x_0$. This method of constructing tangent vectors to a surface is equivalent to the characterization of (3.4).

There is one other property we shall require of coordinate patches, something we call *nonsingularity*: At every $x_0$ in the coordinate patch $x$, if $x_0 = x(t_0)$, then we require that the vectors $\partial_{e_1}x(t_0), \ldots, \partial_{e_p}x(t_0)$ be linearly independent. Then the requirement that the parametrization $x$ be *nonsingular* amounts to

$$\frac{\partial x}{\partial \tau_1}(x_0) \wedge \cdots \wedge \frac{\partial x}{\partial \tau_p}(x_0) \ \neq \ 0.$$

If $\mathcal{M}$ were a one-dimensional curve and $x$ was the path of a particle tracing the curve, then this condition would amount to saying that the particle never stops moving. More generally, we are requiring that at every point of $\mathcal{M}$, we have a tangent $p$-parallelepiped $(\partial x/\partial\tau_1) \wedge \cdots \wedge (\partial x/\partial\tau_p)$ with positive $p$-volume. Another way to think of this is that the parametrization $x$ maps "small" regions of positive $p$-volume in $\mathbb{R}^p$ to "small" tangent sets also of positive $p$-volume.

A particularly useful surface is the *cell*. This is because other surfaces can be thought of as being composed of cells that are joined together "nicely."

In what follows, by $\mathcal{I}$ we mean the unit interval $[0, 1]$. By the $k$-dimensional *unit cube* $\mathcal{I}^k$ we mean $\mathcal{I} \times \cdots \times \mathcal{I}$ with $k$ factors.

**Definition 5.** We say that $\mathcal{M}$ is a $\mathcal{C}^r$ $p$-cell (contained in some $\mathbb{R}^n$) provided there exists a $\mathcal{C}^r$ coordinatization (or parametrization) $x : \mathcal{I}^p \to \mathcal{M}$ such that

1. $x : \mathcal{I}^p \to \mathcal{M}$ is one-to-one and onto.
2. $x$ and $x^{-1}$ are $\mathcal{C}^r$.
3. $\frac{\partial x}{\partial\tau_1} \wedge \cdots \wedge \frac{\partial x}{\partial\tau_p} \neq 0$ everywhere on $\mathcal{M}$.

See Figure 3.2 for a picture of what cells look like.

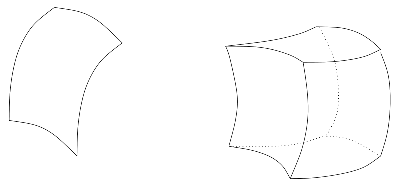

Figure 3.2: $2$-cell and $3$-cell

## 3.4 Tangent blades and orientation

Recall that we can specify an *orientation* for an arc $A$ by attaching a unit tangent vector to every point of $A$ in a continuous fashion. (Figure 3.3.)

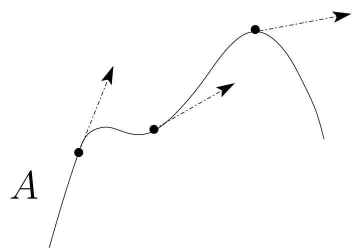


Figure 3.3: Oriented arc

How do we generalize *orientation* to a surface?

If we have coordinatization $x : U \to \mathcal{M}$ of a $p$-surface $\mathcal{M}$ (where $U$ is an open subset of $\mathbb{R}^p$) and this induces coordinates $(\chi_1, \dots, \chi_p)$ on $\mathcal{M}$, then since each $\partial x/\partial \chi_i$ is a tangent vector to $\mathcal{M}$, we can consider

$$\frac{\partial x}{\partial \chi_1} \wedge \cdots \wedge \frac{\partial x}{\partial \chi_p}$$

to be a *tangent $p$-vector* to $\mathcal{M}$. We may visualize it as a parallelepiped that is tangent to $\mathcal{M}$ at the point $x_0 \in \mathcal{M}$ at which each of the partials is evaluated.

A simple $k$-vector $a_1 \wedge \cdots \wedge a_k$ is called a *blade* provided it is not $0$. Our requirement of a parametrization $x$ that

$$\frac{\partial x}{\partial \chi_1} \wedge \cdots \wedge \frac{\partial x}{\partial \chi_p} \neq 0$$

then amounts to the claim that this tangent parallelepiped is a $p$-blade.

By an *orientation* of the $p$-surface $\mathcal{M}$, we mean a continuous function $w$ that assigns to every point $x$ of $\mathcal{M}$ a unit tangent blade $w(x)$. That is, $w(x)$ must have the form $a_1 \wedge \cdots \wedge a_p$ where each $a_i$ is a tangent vector to $\mathcal{M}$ at $x$ and $|w(x)| = 1$.

**Example 6.** By the *standard orientation* of $\mathbb{R}^n$ we mean the constant $n$-blade $e_1 \cdots e_n = e_1 \wedge \cdots \wedge e_n$.

Every $\mathcal{C}^1$ surface $\mathcal{M}$ has local *orientations* around every point $x_0 \in \mathcal{M}$. That is, a unit tangent blade $w(x)$ can be assigned continuously to some neighborhood of $x_0$ in $\mathcal{M}$. If the orientation can be extended to all of $\mathcal{M}$, the surface is *orientable*; otherwise, as in the case of a Möbius strip, the surface is *nonorientable*.

A connected, orientable surface $\mathcal{M}$ has precisely two orientations. If one of them is $w$, the other is $-w$.

If $w$ is an orientation of the $p$-surface $\mathcal{M}$ and $x$ is a parametrization of $\mathcal{M}$, we say that $x$ *agrees* with the orientation of $\mathcal{M}$ provided

$$w = \frac{\frac{\partial x}{\partial \chi_1} \wedge \cdots \wedge \frac{\partial x}{\partial \chi_p}}{\left|\frac{\partial x}{\partial \chi_1} \wedge \cdots \wedge \frac{\partial x}{\partial \chi_p}\right|}.$$

## 3.5 Integrals

We start with the change-of-variables formula for integrals:

Suppose $U$ and $V$ are open subsets of $\mathbb{R}^n$ and $x : U \to V$ is a one-to-one onto map such that both $x$ and $x^{-1}$ are $\mathcal{C}^1$. If $\phi : V \to \mathbb{R}$ and $\phi$ is integrable over $V$, then $(\phi \circ x)\,|\det\, x'|$ is integrable over $U$ and

$$\int_U (\phi \circ x)\,|\det\, x'| = \int_V \phi. \tag{3.5}$$

(See [4] or [10] for a proof.) The expression $|\det\, x'|$ is a kind of "local magnification factor" for the way the map $x$ changes $n$-dimensional volume at any given point. The expression $\det\, x'$ may be familiar to the reader as the *Jacobian determinant*.

Using the wedge product, this formula can be generalized to define what we mean by an integral over a $k$-dimensional surface $\mathcal{M}$ in $\mathbb{R}^n$.

Suppose that $x : U \to \mathcal{M}$ is a parametrization of $\mathcal{M}$ and it assigns coordinates $(\tau_1, \ldots, \tau_k)$ to points of $\mathcal{M}$. Recall this means that a point $p \in \mathcal{M}$ has coordinates $(\tau_1, \ldots, \tau_k)$ with respect to the parametrization (or coordinate patch) $x$ if $p = x(\tau_1, \ldots, \tau_k)$. We intend $U$ to be a "nice" subset of $\mathbb{R}^k$, for example, a $k$-cell, rectangle, simplex, open set, etc. We set

$$\int_{\mathcal{M}} \phi \stackrel{\text{def.}}{=} \int_U (\phi \circ x) \left| \frac{\partial x}{\partial \tau_1} \wedge \cdots \wedge \frac{\partial x}{\partial \tau_k} \right|. \tag{3.6}$$

The integral on the right is that of a real-valued function over a "nice" subset of $\mathbb{R}^k$, so we expect it to be integrable using standard techniques of calculus.

We know that

$$\frac{\partial x}{\partial \tau_1} \wedge \cdots \wedge \frac{\partial x}{\partial \tau_k}$$

can be envisioned as an oriented $k$-dimensional parallelepiped that is tangent to $\mathcal{M}$, and

$$\left| \frac{\partial x}{\partial \tau_1} \wedge \cdots \wedge \frac{\partial x}{\partial \tau_k} \right|$$

is the $k$-volume of that parallelepiped. In the case where $k = n$, then

$$\left| \frac{\partial x}{\partial \tau_1} \wedge \cdots \wedge \frac{\partial x}{\partial \tau_k} \right| = |\det(x')|,$$

and we are back to the change-of-variables formula.

**Example 7.** Let $S^1$ be the unit circle in $\mathbb{R}^2$. We will take $\mathcal{M}$ to be $S^1 \times S^1$. This is a torus in $\mathbb{R}^4$ and satisfies the two equations $\chi_1^2 + \chi_2^2 = 1$ and $\chi_3^2 + \chi_4^2 = 1$. We know the length of $S^1$ is $2\pi$. Since $\mathcal{M}$ is a cartesian product with the two copies of $S^1$ lying in orthogonal spaces, it seems reasonable to guess that the area of $\mathcal{M}$ (the 2-dimensional volume) should be a product of lengths, $(2\pi)^2 = 4\pi^2$. At the same time, one would expect the 2-volume of $\mathcal{M}$ to be given by $\int_{\mathcal{M}} 1$. Do the two intuitions agree? Let us figure out how to evaluate the integral.

We know that we can parametrize $S^1$ thus:

$$\theta \mapsto \cos(\theta)\, e_1 + \sin(\theta)\, e_2.$$

So we take

$$x(\psi, \xi) = \cos(\psi)\, e_1 + \sin(\psi)\, e_2 + \cos(\xi)\, e_3 + \sin(\xi)\, e_4,$$

where $(\psi, \xi) \in [0, 2\pi] \times [0, 2\pi]$, as our parametrization of $\mathcal{M}$.

(There is a slight technical difficulty here since this parametrization is not one-to-one on the edges of the square. However the problem occurs on a "small set," one that is of "measure zero," and it can be shown that this does not influence the answer.)

We see that

$$\begin{aligned} \frac{\partial x}{\partial \psi} &= -\sin(\psi)\, e_1 + \cos(\psi)\, e_2, \\ \frac{\partial x}{\partial \xi} &= -\sin(\xi)\, e_3 + \cos(\xi)\, e_4, \end{aligned}$$

and we readily calculate that

$$\Big| \frac{\partial x}{\partial \psi} \wedge \frac{\partial x}{\partial \xi} \Big| = 1.$$

We finally evaluate the integral:

$$\begin{aligned} \int_{\mathcal{M}} 1 &= \int_{[0,2\pi]^2} \Big| \frac{\partial x}{\partial \psi} \wedge \frac{\partial x}{\partial \xi} \Big| \\ &= \int_0^{2\pi} \int_0^{2\pi} 1 \, d\psi \, d\xi = 4\pi^2. \end{aligned}$$

We can replace $\phi$ in the integral of (3.6) by a $p$-vector field $f$. We need only start with the standard basis $\{e_i\}_{i=1}^n$ for $\mathbb{R}^n$ and write $f$ in terms of it:

$$f = \sum_{|I|=p} \phi_I e_I$$

where it is understood that $I$ ranges over all multi-indices which are both ordered and of length $p$ and each $\phi_I$ is a real-valued function. $f$ always has such a decomposition, and it is unique. We then set

$$\int_{\mathcal{M}} f \overset{\text{def.}}{=} \sum_{|I|=p} \Big( \int_{\mathcal{M}} \phi_I \Big)\, e_I. \tag{3.7}$$

Of course $\{e_i\}_{i=1}^n$ can be replaced by any basis $\{u_i\}_{i=1}^n$ for $\mathbb{R}^n$ provided the basis elements are constant vectors. We call (3.6) and (3.7) *unoriented integrals*.

We know that in elementary, single-variable calculus, we deal with *oriented* integrals. We can see this from the fact that given a real-valued function of a single variable, $\phi(\tau)$, we have

$$\int_{\alpha}^{\beta} \phi(\tau)\, d\tau \;=\; -\int_{\beta}^{\alpha} \phi(\tau)\, d\tau.$$

That is, orientation is taken account of by noting the direction in which we integrate.

We shall see later how to generalize the notion of *oriented integral* to surfaces. It is convenient to first develop the ideas of the *geomtric product* and *geometric algebra*.

# Chapter 4

# Geometric Algebra

## 4.1 $\mathbb{G}^n$ and grades

We now construct the *geometric algebra over* $\mathbb{R}^n$ and denote it $\mathbb{G}^n$. This is a vector space that contains and is generated by $\mathbb{R}$, $\mathbb{R}^n$, $\Lambda^2\mathbb{R}^n$, …, $\Lambda^n\mathbb{R}^n$. In $\mathbb{G}^n$ it is perfectly legal to add $k$-vectors which have different values of $k$. Thus we get to write expressions such as $\sqrt{2} - 7\, e_1 \wedge e_3$. It follows that in $\mathbb{G}^3$, for example, every element will be of the form $\alpha + a + b + c$ where $\alpha$ is a real number, $a$ is a vector, $b$ is a 2-vector, and $c$ is a 3-vector. We call the elements of $\mathbb{G}^n$ *multivectors*.

We can write $\mathbb{G}^n$ as a direct sum of the different spaces of $k$-vectors in $\mathbb{R}^n$:

$$\mathbb{G}^n \;=\; \mathbb{R} \oplus \mathbb{R}^n \oplus \Lambda^2\mathbb{R}^n \oplus \cdots \oplus \Lambda^n\mathbb{R}^n.$$

Of course, we feel free to write $\mathbb{R} = \Lambda^0\mathbb{R}^n$ and $\mathbb{R}^n = \Lambda^1\mathbb{R}^n$. If a multivector is a sum of $k$-vectors, then we say it is of *grade* $k$. Thus $4\, e_1 \wedge e_2 - \pi\, e_2 \wedge e_3$ is a grade 2 multivector. On the other hand, $\sqrt{2} - 7\, e_1 \wedge e_3$ cannot be assigned a unique grade because it is the sum of a grade 0 and a grade 2 term. If $a$ is a multivector, we use the symbol $\langle a \rangle_k$ for the sum of the grade $k$ terms that occur in it. Thus for every multivector $a$ in $\mathbb{G}^n$ we have a unique decomposition:

$$a \;=\; \langle a \rangle_0 + \langle a \rangle_1 + \cdots \langle a \rangle_n.$$

If we take $a = \sqrt{2} - 7\, e_1 \wedge e_3$, an element of $\mathbb{G}^3$, then for this multivector,

$$\langle a \rangle_0 \;=\; \sqrt{2},$$
$$\langle a \rangle_1 \;=\; 0,$$

$$\langle a \rangle_2 = -7\, e_1 \wedge e_3,$$
$$\langle a \rangle_3 = 0.$$

## 4.2 The geometric product

In spaces of $k$-vectors, we are familiar with the operations of multiplication by scalars, addition, the dot product, and the wedge product. These operations can be extended into $\mathbb{G}^n$. And we add one more operation, the *geometric product*, which can be thought of as a generalization of both the dot and the wedge products.

We write the geometric product of multivectors $a$ and $b$ as $ab$. This is not, in general, commutative. Careful constructions of it are given in [7] and [12]. Here we follow an exposition reminiscent of [6] and simply describe the properties of the geometric product.

**Properties of the geometric product**

1. If $a, b, c \in \mathbb{G}^n$ and $\lambda \in \mathbb{R}$, then
   (a) $1\, a = a\, 1 = a$.
   (b) $a\,(b + c) = ab + ac$ and $(a + b)\, c = ac + bc$.
   (c) $\lambda\,(ab) = (\lambda a)\, b = a\,(\lambda b)$.
   (d) $a(bc) = (ab)c$.
2. If $a, b \in \mathbb{R}^n$, then
$$ab = a \bullet b + a \wedge b.$$
3. If $u_1, \ldots, u_p$ are orthogonal vectors, then
$$u_1 \cdots u_p = u_1 \wedge \cdots \wedge u_p.$$

Notice that it is properties 2 and 3 that exhibit the connection between the geometric, dot, and wedge products.

An immediate consequence of property 2 is that if $a$ is a vector of $\mathbb{R}^n$, then $aa = |a|^2$. This is because $a \bullet a = |a|^2$ and $a \wedge a = 0$.

If we wish to calculate a geometric product, it is often helpful to express our multivectors in terms of an orthonormal basis for $\mathbb{R}^n$. Suppose, for example, that $u_1, u_2, u_3$ are orthonormal vectors. Then

$$\begin{aligned} u_1(u_1 u_2) &= u_2 \quad \text{(because } |u_1|^2 = 1\text{)}, \\ u_2(u_1 u_3) &= u_2 \wedge u_1 \wedge u_3 = -u_1 \wedge u_2 \wedge u_3 = -u_1 u_2 u_3. \end{aligned}$$

We can show the following:

**Proposition 5.** *If $\{u_i\}_{i=1}^n$ is a basis for $\mathbb{R}^n$, then the set consisting of $1$ and the blades $u_{i_1} \wedge \cdots \wedge u_{i_k}$ such that $i_1 < \cdots < i_k$ and $k = 0, 1, \ldots, n$ constitutes a basis for $\mathbb{G}^n$.*

Thus a basis for $\mathbb{G}^3$ is

$$1,\ e_1,\ e_2,\ e_3,\ e_1e_2,\ e_1e_3,\ e_2e_3,\ e_1e_2e_3.$$

Keep in mind when we say this, that a basis element such as $e_1e_2$ is the same thing as $e_1 \wedge e_2$ because $e_1$ and $e_2$ are orthogonal. So we can also say that a basis for $\mathbb{G}^3$ is

$$1,\ e_1,\ e_2,\ e_3, \quad e_1 \wedge e_2, \quad e_1 \wedge e_3, \quad e_2 \wedge e_3, \quad e_1 \wedge e_2 \wedge e_3.$$

**Corollary 1.** *The dimension of $\mathbb{G}^n$ is $2^n$.*

This is seen from the fact that the dimension of $\Lambda^k \mathbb{R}^n$ is $\binom{n}{k}$ and

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

## 4.3 Extending reversion

One of our concerns is to extend operations defined for $k$-vectors to arbitrary multivectors. (Remember, not all multivectors are $k$-vectors. An example is $\sqrt{2} - 7\,e_1e_3$. We want our extended operations to work in this larger space.) The first of these is reversion:

If $u_1, \ldots, u_k$ are orthogonal vectors, then we see that

$$\begin{aligned} (u_1 \cdots u_k)^\dagger &= (u_1 \wedge \cdots \wedge u_k)^\dagger \\ &= u_k \wedge \cdots \wedge u_1 \end{aligned}$$

$$= (-1)^{\frac{k(k-1)}{2}} u_1 \wedge \cdots \wedge u_k$$
$$= (-1)^{\frac{k(k-1)}{2}} u_1 \cdots u_k.$$

We can do this because we know how to take the reversion of a $k$-vector. We know that a multivector in $\mathbb{R}^n$ has a unique expansion into $k$-vectors thus:

$$a \;=\; \langle a \rangle_0 + \langle a \rangle_1 + \cdots + \langle a \rangle_n.$$

We define the *reversion of a multivector* in this way:

$$a^\dagger \;\stackrel{\text{def.}}{=}\; \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle a \rangle_k.$$

One can, of course, show that reversion is linear (that is, $(\lambda a)^\dagger = \lambda a^\dagger$ and $(a+b)^\dagger = a^\dagger + b^\dagger$) and that $(ab)^\dagger = b^\dagger a^\dagger$.

## 4.4 The scalar product

The second operation we extend from $k$-vectors to multivectors is the dot product. There is definitely more than one useful way to do this; four of them are considered in [12]. We exhibit only one, the *scalar product*.

**Definition 6.** If $a$ and $b$ are multivectors in $\mathbb{R}^n$, that is, $a, b \in \mathbb{G}^n$, then their *scalar product* is $\langle ab^\dagger \rangle_0$.

This product is commutative, and it does not matter which factor gets the dagger:

$$\langle ab^\dagger \rangle_0 \;=\; \langle a^\dagger b \rangle_0 \;=\; \langle ba^\dagger \rangle_0 \;=\; \langle b^\dagger a \rangle_0.$$

It is easily seen that the scalar product generalizes the dot product of vectors: Suppose that $a, b \in \mathbb{R}^n$; that is, $a$ and $b$ are grade 1. Note that $b^\dagger = b$ trivially. We know that $ab = a \bullet b + a \wedge b$. We see that $a \bullet b$ is grade 0 while $a \wedge b$ is grade 2. Thus

$$\langle ab^\dagger \rangle_0 \;=\; \langle ab \rangle_0 \;=\; \langle a \bullet b + a \wedge b \rangle_0 \;=\; a \bullet b.$$

It can also be shown that if $a$ and $b$ are simple $k$-vectors, $a = a_1 \wedge \cdots \wedge a_k$ and $b = b_1 \wedge \cdots \wedge b_k$, then

$$\langle ab^\dagger \rangle_0 \;=\; \det(a_i \bullet b_j)_{k \times k},$$

the determinant of a $k \times k$ matrix, so that the scalar product generalizes the dot product of $k$-vectors.

The scalar product looks particularly nice if one writes everything in terms of an orthonormal basis:

Let $\{u_i\}_{i=1}^n$ be an orthonormal basis for $\mathbb{R}^n$. We know that $\{u_I\}_I$, where $I$ runs through all ordered multi-indices, is a basis for $\mathbb{G}^n$. We see that

$$\left\langle u_I u_J^\dagger \right\rangle_0 = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

This follows from the fact that if $u_I$ is a $p$-vector and $u_J$ is a $q$-vector with $p \neq q$, then the product $u_I u_J^\dagger$ cannot be grade $0$. While if $p = q$, then the scalar product reduces to the dot product of $p$-vectors. In any event, we can treat $\{u_I\}_I$ as an "orthonormal" basis for $\mathbb{G}^n$.

Next let $a$ and $b$ be multivectors in $\mathbb{R}^n$ and write their unique expansions in terms of $\{u_I\}_I$:

$$a = \sum_I \alpha_I u_I \quad \text{and} \quad b = \sum_I \beta_I u_I.$$

Then we have

$$\left\langle a a^\dagger \right\rangle_0 = \sum_I \alpha_I^2,$$
$$\left\langle a b^\dagger \right\rangle_0 = \sum_I \alpha_I \beta_J.$$

We define the *magnitude* of the multivector $a$ thus:

$$|a| \stackrel{\text{def.}}{=} \sqrt{\left\langle a a^\dagger \right\rangle_0}.$$

Notice that we are simply echoing things we said earlier about $k$-vectors in Section 2.4 and Proposition 3 but now on the broader stage of $\mathbb{G}^n$.

## 4.5 The wedge product of multivectors

The easiest operation to extend to multivectors is the wedge product. The quick-and-dirty way to do it is to assume that this extended operation has the following, obviously desirable properties:

If $a, b, c$ are multivectors in $\mathbb{R}^n$ and $\lambda$ is scalar, then the following hold:

1. $a \wedge (b+c) \ = \ (a \wedge b) + (a \wedge c)$.
2. $(a+b) \wedge c \ = \ (a \wedge c) + (b \wedge c)$.
3. $\lambda(a \wedge b) \ = \ (\lambda a) \wedge b \ = \ a \wedge (\lambda b)$.
4. $\lambda \wedge a \ = \ a \wedge \lambda \ = \ \lambda a$.
5. $(a \wedge b)^\dagger \ = \ b^\dagger \wedge a^\dagger$.

This permits us to reduce all wedge products of multivectors to wedge products of simple $p$- and $q$-vectors which we already (presumably) know how to do. Thus, for example,

$$\begin{aligned}(2 + 3\, e_1 e_4) \wedge (1 - e_1 + e_2) \ &= \ 2 - 2e_1 + 2e_2 \\ &\quad + 3\, e_1 e_4 - 3\,(e_1 e_4) \wedge e_1 + 3\,(e_1 e_4) \wedge e_2 \\ &= \ 2 - 2e_1 + 2e_2 + 3\, e_1 e_4 - 3 e_1 e_2 e_4.\end{aligned}$$

(Notice in this calculation that

$$\begin{aligned}(e_1 e_4) \wedge e_1 \ &= \ e_1 \wedge e_4 \wedge e_1 \ = \ 0, \\ (e_1 e_4) \wedge e_2 \ &= \ e_1 \wedge e_4 \wedge e_2 \ = \ -e_1 \wedge e_2 \wedge e_4\end{aligned}$$

because the vectors are orthogonal.)

Of course, to simply assume the properties we want is cheating. It can easily lead to disaster (as in, "Let us assume the following circle has three corners ..."). To do this properly, we ought to provide a mathematical proof that the extension can be carried out. We do not do this here, but one way to do it is to use the following odd-looking definition of the extended wedge product:

$$a \wedge b \ \stackrel{\text{def.}}{=} \ \sum_{p,q=0}^{n} \left\langle \langle a \rangle_p \, \langle b \rangle_q \right\rangle_{p+q}.$$

(Details provided in [12].)

## 4.6 Division by multivectors

A very useful property of geometric algebra is that one can always divide by blades; in particular, one can divide by nonzero vectors. This something one cannot do in vector analysis.

Suppose that $a$ is a $k$-blade in $\mathbb{R}^n$. We know that $a \neq 0$ so $|a| > 0$. Notice that since $aa^\dagger = a^\dagger a = |a|^2$, we have

$$a\,\frac{a^\dagger}{|a|^2} \;=\; \frac{a^\dagger}{|a|^2}\,a \;=\; 1.$$

This amounts to saying that

$$a^{-1} \;=\; \frac{a^\dagger}{|a|^2}.$$

A case worthy of note is when we have a unit blade, as, for example, an orientation $w$ of a surface. Then

$$w^{-1} \;=\; w^\dagger \quad \text{because } |w| = 1.$$

Though blades always have inverses (with respect to the geometric product), there are nonzero multivectors that do not. Division is only possible *some* of the time.

Division often has an important geometric interpretation. To see this, let $a$ be a $k$-blade in $\mathbb{R}^n$ and let $e = e_1 \cdots e_n$ where $\{e_i\}_{i=1}^n$ is the standard basis. We know that $e$ is the standard orientation of $\mathbb{R}^n$. It is always possible to find an orthonormal basis $\{u_i\}_{i=1}^n$ of $\mathbb{R}^n$ such that $a = \lambda\, u_1 \cdots u_k$ where $\lambda = |a|$. We know that $u = u_1 \cdots u_n$ must also be an orientation for $\mathbb{R}^n$, and since a connected, $n$-dimensional surface (of which $\mathbb{R}^n$ is an example) can have only two orientations, either $u = e$ or $u = -e$. Let us assume, for the same of simplicity, that $u = e$. Then

$$\begin{aligned} ea^{-1} \;&=\; \frac{1}{\lambda}\,(u_1 \cdots u_n)(u_1 \cdots u_k)^\dagger \\ &= \frac{(-1)^r}{\lambda}\,(u_k \cdots u_1)(u_1 \cdots u_n) \\ &= \frac{(-1)^r}{\lambda}\,(u_{k+1} \cdots u_n) \end{aligned}$$

for some integer $r$. Now $(-1)^r(1/\lambda)(u_{k+1} \cdots u_n)$ is clearly orthogonal to $a = \lambda(u_1 \cdots u_k)$. So what we have shown is that if you divide the orientation of $\mathbb{R}^n$ by a $k$-blade $a$, the result must be an $(n-k)$-blade that is orthogonal to $a$.

## 4.7 Reciprocal vectors

Given a set of linearly independent vectors $\{a_i\}_{i=1}^k$, a very useful trick we can play with the geometric algebra is to construct the *reciprocal set* $\{b_i\}_{i=1}^k$. (We shall demonstrate the usefulness in the next chapter.) The set $\{b_i\}_{i=1}^k$ is unique and is defined by the properties

$$\begin{aligned} &\{a_i\}_{i=1}^k \text{ and } \{b_i\}_{i=1}^k \text{ span the same subspace, and} \\ a_i \bullet b_j \; &= \; \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

Here is a formula for computing $b_j$ from $\{a_i\}_{i=1}^k$: Set $a = a_1 \wedge \cdots \wedge a_k$. Then

$$\begin{aligned} b_j &= (-1)^{i-1}(a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_k)a^{-1} \\ &= (-1)^{k-i}a^{-1}(a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_k), \end{aligned}$$

where $\widehat{a_i}$ means that $a_i$ is omitted in this product. Notice that in this formula, we are taking the geometric product of $a^{-1}$ and $a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_k$.

**Example 8.** Let $V$ be the subspace $\chi_2 = \chi_3$ of $\mathbb{R}^3$. A basis is $a_1 = e_1$ and $a_2 = e_2 + e_3$. We readily compute that

$$\begin{aligned} a \; &= \; a_1 \wedge a_2 \; = \; e_1e_2 + e_1e_3, \\ a^{-1} \; &= \; -\frac{1}{2}(e_1e_2 + e_1e_3), \\ b_1 \; &= \; a_2a^{-1} \; = \; e_1, \\ b_2 \; &= \; -a_1a^{-1} \; = \; \frac{1}{2}(e_2 + e_3). \end{aligned}$$

If $\{a_i\}_{i=1}^k$ is a basis for a subspace $V$, we naturally call $\{b_i\}_{i=1}^k$ the *reciprocal basis*. An independent set of vectors $\{a_i\}_{i=1}^k$ is often called a *frame*. In that case, $\{b_i\}_{i=1}^k$ is called the *reciprocal frame*.

# Chapter 5

# Derivatives and the Fundamental Theorem

## 5.1 Directional derivatives

We assume that $\mathcal{M}$ is $p$-dimensional $\mathcal{C}^1$ surface in $\mathbb{R}^m$.

Suppose that $f$ is a multivector field on $\mathcal{M}$; that is, $f : \mathcal{M} \to \mathbb{G}^m$. We extend our definition of directional derivative to $f$:

$$\partial_u f(x_0) \stackrel{\text{def.}}{=} \lim_{\lambda \to 0} \frac{1}{\lambda}\big(f(x_0 + \lambda u) - f(x_0)\big)$$

where $x_0$ is a point in $\mathcal{M}$, $u$ is a vector in $\mathbb{R}^m$, and, of course, $\lambda$ is a scalar. Provided $f$ is "nice," for example, if it is $\mathcal{C}^1$, then given $x_0$, there is a linear transformation $f'(x_0) : \mathbb{R}^m \to \mathbb{G}^m$ such that

$$\partial_u f(x_0) = \big[f'(x_0)\big](u).$$

The map $u \mapsto \big[f'(x_0)\big](u)$ as the ***differential of*** $f$ ***at*** $x_0$. If we have two maps $f$ and $g$ and can form their composite, then we have the formula

$$\partial_u\big(f \circ g\big)(x_0) = \big[\big(f \circ g\big)'(x_0)\big](u) = \big[f'\big(g(x_0)\big)\big]\,\big[g'(x_0)\big](u) \tag{5.1}$$

which is, in effect, the *chain rule* of multivariable calculus. This is often helpful for establishing theoretical results.

Continuing with our multivector field $f$ on $\mathcal{M}$, suppose we also have a parametrization $x : U \to \mathcal{M}$ which induces coordinates $(\chi_1, \ldots, \chi_p)$ on $\mathcal{M}$.

Then at the point $x_0 \in \mathcal{M}$ we define

$$\frac{\partial f}{\partial \chi_i}(x_0) \overset{\text{def.}}{=} \partial_{e_i}\big(f \circ x\big)(r_0) = \big[\big(f \circ x\big)'(r_0)\big](e_i)$$

where $U$ is a suitable subset of $\mathbb{R}^p$ and $r_0$ is the unique point in $\mathbb{R}^p$ such that $x(r_0) = x_0$. Notice that we are treating $\partial f / \partial \chi_i$ as a function defined on the surface $\mathcal{M}$; this works because $x$ is a one-to-one function.

**Example 9.** Let us take $\mathcal{M}$ to be the paraboloid $\chi_3 = \chi_1^2 + \chi_2^2$ in $\mathbb{R}^3$. (Figure 5.1.) We give a parametrization $x$ that maps the $\chi_1\chi_2$-plane straight up onto

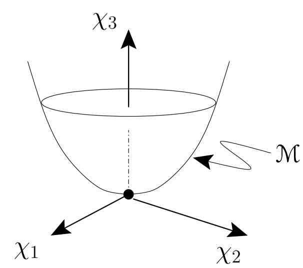


Figure 5.1: Paraboloid

the paraboloid:

$$x(\chi_1, \chi_2) = (\chi_1, \chi_2, \chi_1^2 + \chi_2^2).$$

Let us make up a multivector field $f$ that is defined on $\mathcal{M}$ (and indeed on all of $\mathbb{R}^3$):

$$f(\chi_1, \chi_2, \chi_3) = \cos(\chi_1) + \sin(\chi_2)\, e_3 + \chi_3\, e_1 e_2.$$

Then

$$\big(f \circ x\big)(\chi_1, \chi_2) = \cos(\chi_1) + \sin(\chi_2)\, e_3 + (\chi_1^2 + \chi_2^2)\, e_1 e_2,$$

so

$$\frac{\partial f}{\partial \chi_1} = -\sin(\chi_1) + 2\chi_1\, e_1 e_2,$$
$$\frac{\partial f}{\partial \chi_2} = \cos(\chi_2)\, e_3 + 2\chi_2\, e_1 e_2.$$

**Example 10.** Suppose a parametrization $x$ assigns coordinates $(\chi_1, \ldots, \chi_p)$ to a surface $\mathcal{M}$. It is also often convenient to regard $\chi_i$ not as the $i$th coordinate of a point but as a map $\chi_i : \mathcal{M} \to \mathbb{R}$ that assigns to a point on $\mathcal{M}$ its $i$th coordinate. That is, if $x_0$ is a point on $\mathcal{M}$ and $x(\chi_{01}, \ldots, \chi_{0p}) = x_0$, then $\chi_i(x_0) = \chi_{0i}$. This means that $\chi_i \circ x = \pi_i$, the projection $\mathbb{R}^p \to \mathbb{R}$ that selects the $i$th entry of of a $p$-tuple. It then becomes easy to show that

$$\frac{\partial \chi_i}{\partial \chi_j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It should not be surprising that the following version of the chain rule can be established:

**Proposition 6.** *Let $\mathcal{M}$ be a $\mathcal{C}^1$ surface with two sets of coordinates, $(\chi_1, \ldots, \chi_p)$ and $(\xi_1, \ldots, \xi_p)$. If $f$ is $\mathcal{C}^1$ multivector field on $\mathcal{M}$, then*

$$\frac{\partial f}{\partial \chi_i} = \sum_{j=1}^{p} \frac{\partial f}{\partial \xi_j} \frac{\partial \xi_j}{\partial \chi_i}.$$

## 5.2 The geometric derivative

Let $\mathcal{M}$ be a $\mathcal{C}^1$ $p$-surface in $\mathbb{R}^m$. If $x_0$ is a point in $\mathcal{M}$ and $x$ is a parametrization that induces coordinates $(\chi_1, \ldots, \chi_p)$ on $\mathcal{M}$, then we know from Chapter 3 that $\{(\partial x/\partial \chi_i)(x_0)\}_{i=1}^{p}$ is a basis for the tangent space $T_{x_0}\mathcal{M}$. We know there must be a reciprocal basis; let us designate it $\{d\chi_i(x_0)\}_{i=1}^{p}$. Thus

$$\frac{\partial x}{\partial \chi_i} \bullet d\chi_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

at all points at which the two vector fields can be evaluated.

We now come to the basic operation of geometric calculus.

**Definition 7.** If $f$ is a multivector field on $\mathcal{M}$, then

$$\overrightarrow{\nabla}_{\mathcal{M}} f \stackrel{\text{def.}}{=} \sum_{i=1}^{p} \frac{\partial f}{\partial \chi_i} d\chi_i,$$
$$\overleftarrow{\nabla}_{\mathcal{M}} f \stackrel{\text{def.}}{=} \sum_{i=1}^{p} d\chi_i \frac{\partial f}{\partial \chi_i}.$$

We call these the *right-hand and left-hand geometric derivative* respectively depending on the direction of the arrow.

(In the literature, one usually considers only $\overleftarrow{\nabla}_{\mathcal{M}} f$, and it is called the *vector derivative*; but we prefer the term geometric derivative.)

It is important to note two things about this definition. First, the product of the multivector field $\partial f/\partial\chi_i$ and the vector $d\chi_i$ is the geometric product. In general,

$$\frac{\partial f}{\partial \chi_i}\, d\chi_i \;\neq\; d\chi_i\, \frac{\partial f}{\partial \chi_i},$$

so the right-hand and left-hand geometric derivatives need not be equal. This is not a real difficulty since one can show that

$$\big(\overrightarrow{\nabla}_{\mathcal{M}} f\big)^{\dagger} \;=\; \overleftarrow{\nabla}_{\mathcal{M}}(f^{\dagger}).$$

The second important point is that the geometric derivative is independent of the choice of coordinates. Thus if $(\xi_1, \ldots, \xi_p)$ is a second set of coordinates on $\mathcal{M}$, we have

$$\sum_{i=1}^{p} \frac{\partial f}{\partial \chi_i}\, d\chi_i \;=\; \sum_{j=1}^{p} \frac{\partial f}{\partial \xi_j}\, d\xi_j.$$

**Example 11.** Take $\mathcal{M}$ to be $\mathbb{R}^m$ and let the parametrization $x$ be the identity map, $x(\chi_1, \ldots, \chi_p) = (\chi_1, \ldots, \chi_p)$. Then $\partial x/\partial\chi_i = e_i$ and $d\chi_i = e_i$. If $\phi$ is a real-valued function on $\mathbb{R}^m$, then

$$\overrightarrow{\nabla}_{\mathcal{M}}\phi \;=\; \overleftarrow{\nabla}_{\mathcal{M}}\phi \;=\; \sum_{i=1}^{m} \frac{\partial \phi}{\partial \chi_i}\, e_i$$

which is simply the gradient of $\phi$.

The following result should not be surprising:

**Proposition 7.** *Suppose $(\chi_1, \ldots, \chi_p)$ and $(\xi_1, \ldots, \xi_p)$ are two sets of coordinates on the $\mathcal{C}^1$ surface $\mathcal{M}$. Then*

$$d\chi_i \;=\; \sum_{j=1}^{p} \frac{\partial \chi_i}{\partial \xi_j}\, d\xi_j.$$

## 5.3 Oriented integrals

Suppose $f$ is a multivector field defined over a surface $\mathcal{M}$. If $x : U \to \mathcal{M}$ is a parametrization of $\mathcal{M}$ (where $U \subseteq \mathbb{R}^p$) we know from Chapter 3 how to integrate $f$ over $\mathcal{M}$:

$$\int_{\mathcal{M}} f \;=\; \int_U (f \circ x) \left| \frac{\partial x}{\partial \chi_1} \wedge \cdots \wedge \frac{\partial x}{\partial \chi_p} \right|. \tag{5.2}$$

The last expression is evaluated using standard calculus-type techniques and will produce a number or a multivector.

Suppose that $\mathcal{M}$ also has an orientation $w$. By an *oriented integral of $f$ over $\mathcal{M}$* we mean something of (approximately) the form

$$\int_{\mathcal{M}} fw$$

where $fw$ is the geometric product of $f$ and $w$ and the integral is evaluated using (5.2). This is a generalization of the oriented integral

$$\int_\alpha^\beta \phi(\chi)\, d\chi$$

of single-variable calculus. Since we are integrating along the interval $[\alpha, \beta]$ which is a set of real numbers, the only orientations are the real numbers $+1$ or $-1$. In this setting, $\int_{\mathcal{M}} fw$ amounts to

$$\int_{[\alpha,\beta]} \phi\,(+1) \quad \text{or} \quad \int_{[\alpha,\beta]} \phi\,(-1)$$

depending on whether we want $\int_\alpha^\beta \phi$ or $\int_\beta^\alpha \phi$.

There are a number of variations possible on the form $\int_{\mathcal{M}} fw$, for example,

$$\int_{\mathcal{M}} wf, \quad \int_{\mathcal{M}} fw^\dagger, \quad \int_{\mathcal{M}} f\,(-w), \quad \text{etc.}$$

We will refer to them all as oriented integrals of $f$. Because of the complexity possible with multivector fields, unlike real-valued single-variable integrals, these may reduce to more than just two values. The particular oriented integral we want will depend on the problem we are trying to solve.

Here is a particularly useful example of an oriented integral:

**Proposition 8.** *Suppose that the* $\mathcal{C}^1$ *surface* $\mathcal{M}$ *has a parametrization* $x : U \to \mathcal{M}$ *which induces coordinates* $(\chi_1, \ldots, \chi_p)$. *Suppose further that* $\mathcal{M}$ *has an orientation* $w$ *that agrees with the parametrization in the sense that*

$$w \ = \ \lambda \Big( \frac{\partial x}{\partial \chi_1} \wedge \cdots \wedge \frac{\partial x}{\partial \chi_p} \Big)$$

*where* $\lambda > 0$. *If* $f$ *is the multivector field* $\phi\,(d\chi_1 \wedge \cdots \wedge d\chi_p)$ *where* $\phi$ *is a real-valued function on* $\mathcal{M}$, *then*

$$\int_{\mathcal{M}} \phi\,(d\chi_1 \wedge \cdots \wedge d\chi_p)\, w^\dagger \ = \ \int_U \big(\phi \circ x\big)(\chi_1, \ldots, \chi_p)\; d\chi_1 \cdots d\chi_p$$

*where the last integral is an iterated integral, and each* $d\chi_i$ *is not a vector but instead indicates the variable with respect to which one is integrating.*

Of course, in this proposition,

$$\lambda \ = \ \frac{1}{\left| \frac{\partial x}{\partial \chi_1} \wedge \cdots \wedge \frac{\partial x}{\partial \chi_p} \right|}.$$

The integration formula is very straightforward to establish.

## 5.4 Induced orientation

In vector analysis, one may talk of using the unit outward normal vector to the boundary of a region to define an orientation of that boundary. (Figure 5.2.) In a higher dimensional setting, it is not clear one can define analogous concepts. How, for example, would one do this for the boundary of a $k$-dimensional cell in $\mathbb{R}^n$? Geometric algebra is very helpful here.

In general, if $\mathcal{M}$ is a $p$-cell in $\mathbb{R}^m$ with orientation $w$, we would expect an orientation $\partial w$ of the boundary $\partial\mathcal{M}$ of the cell to be a unit tangent $(p-1)$-blade which is continuous (except where one slips from one face to another of the cell). (Figure 5.3.) We assume here that $p \geq 2$ and that $\mathcal{M}$ is at least $\mathcal{C}^1$.

Assume $\mathcal{M}$ is a cell with orientation $w$. It turns out that if one knows either the unit outward normal vector $n$ to the boundary $\partial\mathcal{M}$ at a given point or the orientation of $\partial\mathcal{M}$ at that point, then one can define the other concept; further, this can be done in two different, useful ways.

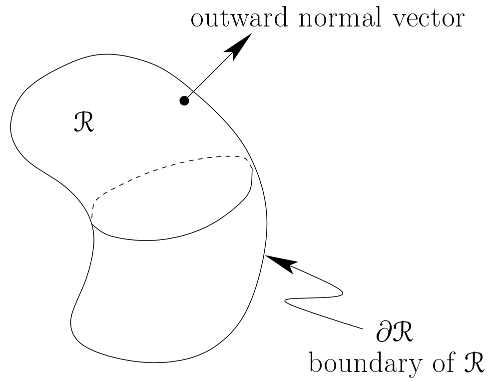


Figure 5.2: A region $\mathcal{R}$ with outward normal vector from the boundary

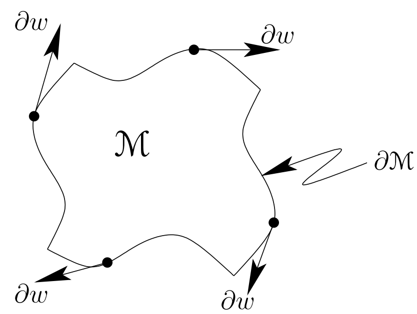


Figure 5.3: Cell with oriented boundary

The trick is that these quantities should satisfy the equations

$$\overrightarrow{\partial} w \;=\; nw \quad \text{and} \quad \overleftarrow{\partial} w \;=\; wn.$$

$\overrightarrow{\partial} w$ and $\overleftarrow{\partial} w$ are, respectively, the *right-hand* and *left-hand induced orientation* on $\partial\mathcal{M}$, and $n$ is the unit outward normal vector to the boundary of $\mathcal{M}$. In some dimensions, these two induced orientations coincide, and in others, they have opposite signs. They are connected by the equation

$$\left(\overrightarrow{\partial} w\right)^{\dagger} \;=\; \overleftarrow{\partial}\left(w^{\dagger}\right).$$

**Example 12.** Let the unit square $\mathcal{I}^2$ in $\mathbb{R}^2$ have orientation $w = e_1e_2$. We see in Figure 5.4 that we should have $n = e_1$ on the right side of $\partial\mathcal{I}^2$, $n = -e_1$ on

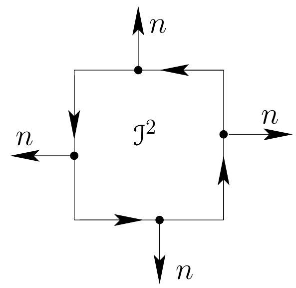


Figure 5.4: Induced orientation on $\partial\mathcal{I}^2$

the left side, and so forth. For the right side of the boundary, since $n = e_1$, we have $\vec{\partial} w = e_2$. For the top side, since $n = e_2$, we see that $\vec{\partial} w = -e_1$. And so forth. That is, $\vec{\partial} w$ amounts to giving $\partial\mathcal{M}$ the counterclockwise orientation.

When dealing with a $p$-cell $\mathcal{M}$, one can start with an induced orientation on $\partial I^p$ and use it to construct an induced orientation on $\partial\mathcal{M}$.

**Example 13.** Suppose $\mathcal{M}$ is a 2-cell with a parametrization $x : \mathcal{I}^2 \to \mathcal{M}$. Let $\mathcal{I}^2$ have the orientation $e_1 e_2$, and the orientation of $\partial\mathcal{I}^2$ be the counterclockwise one. (Figure 5.5.) We can transfer the orientation of $\mathcal{I}^2$ to an orientation $w$

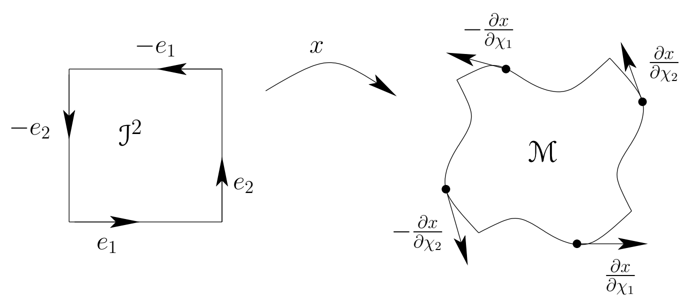


Figure 5.5: Transferring orientation from $\partial I^2$ to $\partial\mathcal{M}$

of $\mathcal{M}$ by the transformation

$$e_1 \wedge e_2 \mapsto \big([x'(r_0)]e_1\big) \wedge \big([x'(r_0)]e_2\big) \;=\; \frac{\partial x}{\partial \chi_1}(x_0) \wedge \frac{\partial x}{\partial \chi_2}(x_0)$$

where $x_0 = x(r_0)$. That is, we take $w$ to be

$$w \;=\; \lambda\Big(\frac{\partial x}{\partial \chi_1} \wedge \frac{\partial x}{\partial \chi_2}\Big)$$

where $\lambda$ is a scalar chosen to make $|w| = 1$. In the same way, one can transfer the tangent 1-blades on $\partial I^2$ to tangent 1-blades on $\partial\mathcal{M}$ via the transformation

$$e_i \mapsto \frac{\partial x}{\partial \chi_i}.$$

Thus, for example, on the right-hand edge of $\partial\mathcal{M}$ (see Figure 5.5), we will have

$$\vec{\partial} w \;=\; \alpha\,\frac{\partial x}{\partial \chi_2}$$

where $\alpha$ is a scalar chosen to normalize $\vec{\partial} w$.

It is straightforward to generalize the discussion of this example to $p$-cells and show how to transfer an induced orientation on $\partial I^p$ to an induced orientation on $\partial\mathcal{M}$.

## 5.5 The Fundamental Theorem

We now state—without proof—a version of the Fundamental Theorem of geometric calculus:

**Theorem 1.** *Let $\mathcal{M}$ be a $\mathcal{C}^2$ $p$-cell with orientation $w$. If $f$ is a $\mathcal{C}^1$ multivector field on $\mathcal{M}$, then*

$$\int_{\partial\mathcal{M}} f\,(\vec{\partial} w) \;=\; \int_{\mathcal{M}} \big(\vec{\nabla}_{\mathcal{M}} f\big)\, w.$$

A second version has the same hypothesis and the equation

$$\int_{\partial\mathcal{M}} (\overleftarrow{\partial} w)\, f \;=\; \int_{\mathcal{M}} w\big(\overleftarrow{\nabla}_{\mathcal{M}} f\big)$$

and readily follows from the fact that

$$\Big(\int_{\mathcal{N}} g\Big)^{\dagger} \;=\; \int_{\mathcal{N}} g^{\dagger}.$$

There is a third version involving two functions $f$ and $g$ on $\mathcal{M}$, but we do not discuss it.

**Example 14.** The single variable form of the Fundamental Theorem from calculus can be viewed as a special case of Theorem 1: Let $\mathcal{M}$ be the interval $[\alpha, \beta] \subseteq \mathbb{R}$ and $f$ be a real-valued function $\phi$ on $[\alpha, \beta]$. We take the orientation of $[\alpha, \beta]$ to be the real number $1$. If we think of $\phi$ as a function of the coordinate $\tau$, then $d\tau = 1$ because our tangent space at each point is $\mathbb{R}$. Then

$$\int_{\mathcal{M}} (\vec{\nabla}_{\mathcal{M}} f)\, w \;=\; \int_{\alpha}^{\beta} \phi'.$$

The boundary of $[\alpha, \beta]$ is a two-point set, $\{\alpha, \beta\}$. It is reasonable to regard the induced orientation of the boundary of $[\alpha, \beta]$ as $-1$ at $\alpha$ and $+1$ at $\beta$ and to interpret the "integral" over a two-point set as

$$\int_{\partial\mathcal{M}} f\,\big(\vec{\partial} w\big) \;=\; f(\beta) - f(\alpha).$$

Thus Theorem 1 yields

$$\int_{\alpha}^{\beta} \phi' \;=\; \phi(\beta) - \phi(\alpha).$$

**Example 15.** Gauss's divergence theorem in $\mathbb{R}^3$ is

$$\int_{\partial\mathcal{M}} f \bullet N \;=\; \int_{\mathcal{M}} \operatorname{div} f$$

where $f$ is a vector field, $\mathcal{M}$ is a bounded $3$-dimensional region, and $N$ is a unit vector directed outward from and orthogonal to the boundary. If we write $f$ in the form $f = \sum_{i=1}^{3} \phi\, e_i$, then

$$\operatorname{div} f \;=\; \sum_{i=1}^{3} \frac{\partial \phi_i}{\partial \chi_i}.$$

This result is easily proved in generalized form in $\mathbb{R}^n$ using the Fundamental Theorem of geometric calculus.

Let $\mathcal{M}$ be a $\mathcal{C}^2$ $n$-cell in $\mathbb{R}^n$, and let us give it the orientation $w = e_1 \cdots e_n$, the same as the standard orientation of $\mathbb{R}^n$. If $N$ is the unit outward normal vector to $\partial\mathcal{M}$ and $\vec{\partial} w$ is the right-hand induced orientation on the boundary of $\mathcal{M}$, they will be related by the equation $\vec{\partial} w = N\, e_1 \cdots e_n$. Now suppose we have a vector field $f$ on $\mathcal{M}$ and we write it in the form $f = \sum_{i=1}^{n} \phi_i e_i$. We take the divergence of the field to be

$$\operatorname{div} f \overset{\text{def.}}{=} \sum_{i=1}^{n} \frac{\partial \phi_i}{\partial \chi_i}.$$

Next we write down the Fundamental Theorem as applied to these objects:

$$\int_{\partial \mathcal{M}} f \, \vec{\partial} w \;=\; \int_{\mathcal{M}} \left(\vec{\nabla}_{\mathcal{M}} f\right) w. \tag{5.3}$$

We rewrite (5.3) as

$$\int_{\partial \mathcal{M}} f \, N \,(e_1 \cdots e_n) \;=\; \int_{\mathcal{M}} \left(\vec{\nabla}_{\mathcal{M}} f\right) e_1 \cdots e_n,$$

and since $e_1 \cdots e_n$ is a constant and in geometric algebra we can divide by blades, we reduce (5.3) to

$$\int_{\partial \mathcal{M}} f \, N \;=\; \int_{\mathcal{M}} \vec{\nabla}_{\mathcal{M}} f. \tag{5.4}$$

We recall that if $(\chi_1, \ldots, \chi_n)$ are just the coordinates of a point in $\mathbb{R}^n$, then $d\chi_i = e_i$. Next we turn the crank and calculate that

$$\vec{\nabla}_{\mathcal{M}} f \;=\; \sum_{i=1}^{n} \frac{\partial f}{\partial \chi_i} e_i \;=\; \operatorname{div} f + \sum_{j<i} \left( \frac{\partial \phi_j}{\partial \chi_i} - \frac{\partial \phi_i}{\partial \chi_j} \right) e_j e_i$$

where $\sum_{j<i}$ mean that we sum over all indices $j$ and $i$ such that $j < i$. Now notice that since $f$ and $N$ are vectors, we have $fN = f \bullet N + f \wedge N$. If we substitute back into (5.4), we obtain

$$\int_{\partial \mathcal{M}} f \bullet N + f \wedge N \;=\; \int_{\mathcal{M}} \operatorname{div} f + \sum_{j<i} \left( \frac{\partial \phi_j}{\partial \chi_i} - \frac{\partial \phi_i}{\partial \chi_j} \right) e_j e_i. \tag{5.5}$$

If we equate the integrals of grade $0$ and grade $2$ in (5.5), we obtain

$$\int_{\partial \mathcal{M}} f \bullet N \;=\; \int_{\mathcal{M}} \operatorname{div} f \quad \text{and} \tag{5.6}$$

$$\int_{\partial \mathcal{M}} f \wedge N \;=\; \sum_{j<i} \int_{\mathcal{M}} \left( \frac{\partial \phi_j}{\partial \chi_i} - \frac{\partial \phi_i}{\partial \chi_j} \right) e_j e_i. \tag{5.7}$$

The first of these equations is the promised generalization of Gauss's divergence theorem. The second one does not fit comfortably in vector analysis since it involves $2$-vectors. However with a little artful trickery, if we set $n = 3$, we may obtain a vector identity involving the cross product.

The Fundamental Theorem can also be used to obtain Green's theorem in the plane, Stokes theorem in $\mathbb{R}^3$, and the generalized Stokes theorem of differential form theory.

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