The Wedge Product and Analytic Geometry

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1. INTRODUCTION. In previous issues of this MONTHLY, Quadrat, Lasserre, and Hiriat-Urruty [**6**] generalized the Pythagorean theorem to orthogonal n-simplexes and Nash [**5**], taking a somewhat different approach, derived the n-dimensional version of the parallelogram law. We would like to argue that a more convenient, uniform, and, indeed, intuitive platform for the derivation of these results and of similar geometric results is provided by the machinery of wedge products and k-vectors.

To do this, we show that it is possible to introduce simple k-vectors—those of the form $a_1 \wedge \cdots \wedge a_k$ where $a_1, \dots, a_k$ are vectors in some $\mathbb{R}^n$—as geometric objects. In calculus or an introductory physics class, one sometimes hears a vector described as an equivalence class of directed line segments. It turns out that one can, in a similar way, define a simple k-vector as an equivalence class of oriented parallelepipeds. We shall show that this geometric definition leads to the standard properties of exterior algebra.

To illustrate the use and convenience of this machinery, we then generalize the triangle law of vector addition and the law of cosines to n-simplexes and rederive the n-dimensional versions of the parallelogram law and the Pythagorean theorem.

2. MATRICES. In all that follows, we restrict ourselves to vectors in $\mathbb{R}^n$ and make essential use of the dot product: recall that if $a, b \in \mathbb{R}^n$ with $a = (\alpha_1, \dots, \alpha_n)$ and $b = (\beta_1, \dots, \beta_n)$, then $a \cdot b = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n$.

It will frequently be useful to talk about an $n \times k$ matrix $A = (a_1, \dots, a_k)$ where each $a_i \in \mathbb{R}^n$. The matrix entries are not really specified without first giving a basis for the space. Thus, if we have in mind the basis $\{f_1, \dots, f_n\}$ for $\mathbb{R}^n$ and we can write $a_i = \alpha_{1i} f_1 + \cdots + \alpha_{ni} f_n$ for each i, then we mean the matrix

$$A = (a_1, \dots, a_k) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1k} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2k} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nk} \end{pmatrix}.$$

We do not always use the standard basis for $\mathbb{R}^n$. It is often convenient to use a basis $\{f_1, \dots, f_n\}$ for $\mathbb{R}^n$ having the property that $\{f_1, \dots, f_k\}$ is a basis for $\operatorname{span}\{a_1, \dots, a_k\}$, in which case

$$A = (a_1, \dots, a_k) = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1k} \\ \vdots & & \vdots \\ \alpha_{k1} & \dots & \alpha_{kk} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

We may then talk about the matrix

$$A' = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1k} \\ \vdots & & \vdots \\ \alpha_{k1} & \dots & \alpha_{kk} \end{pmatrix}$$

associated with the vector subspace span$\{a_1, \dots, a_k\}$, and, by an abuse of notation, we write $A' = (a_1, \dots, a_k)$ even though A and A' are different size matrices.

We will always write our matrices with respect to orthonormal bases because of an important connection between matrix multiplication and the dot product: if $A = (a_1, \dots, a_k)$ and $B = (b_1, \dots, b_k)$, where each a_i and b_j belongs to $\mathbb{R}^n$, and the matrices are specified with respect to an orthonormal basis for $\mathbb{R}^n$, then

$$B^T A = \begin{pmatrix} b_1 \cdot a_1 & \dots & b_1 \cdot a_k \\ \vdots & & \vdots \\ b_k \cdot a_1 & \dots & b_k \cdot a_k \end{pmatrix}$$

where B^T is, of course, the transpose of the $n \times k$ matrix B. A nice feature of this last matrix is that it is independent of the choice of basis. We also feel free to write it as $B^T A = (b_i \cdot a_j)$ or $(b_i \cdot a_j)_{k \times k}$.

Note that if $\{a_1, \dots, a_k\}$ or $\{b_1, \dots, b_k\}$ is linearly dependent, then $\det(a_i \cdot b_j) = 0$. Because of this, in what follows, we shall usually only deal with the case where $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ are sets of linearly independent vectors.

3. SIMPLE *k*-VECTORS.

Parallelepipeds. A parallelogram $\mathcal{A}$ in $\mathbb{R}^n$ is specified by a base point P and two vectors $a_1, a_2 \in \mathbb{R}^n$ which we can think of as "edges" (see Figure 1). The points of $\mathcal{A}$ are those of the form $P + \tau_1 a_1 + \tau_2 a_2$ where τ_1 and τ_2 are arbitrary scalars such that $0 \le \tau_i \le 1$. The 1-dimensional faces of $\mathcal{A}$ are obtained by setting one τ_i equal to 0 or 1 and permitting the other τ_j to range arbitrarily over the unit interval. Thus one example of a 1-dimensional face of $\mathcal{A}$ consists of points of the form $P + \tau_1 a_1$ and another of points of the form $P + a_1 + \tau_2 a_2$. The 0-dimensional faces (or vertices) of $\mathcal{A}$ are found by setting both τ_1 and τ_2 equal to 0 or 1. In this example, the vertices are

$$P, \quad P + a_1, \quad P + a_2, \quad \text{and} \quad P + a_1 + a_2.$$

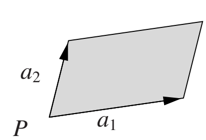


Figure 1. A parallelogram.

More generally, a *k-dimensional parallelepiped* $\mathcal{A}$ in $\mathbb{R}^n$ with base point $P \in \mathbb{R}^n$ and "edges" $a_1, a_2, \dots, a_k$, vectors in $\mathbb{R}^n$, consists of all points of the form

$$P + \tau_1 a_1 + \tau_2 a_2 + \dots + \tau_k a_k \quad \text{where} \quad 0 \le \tau_1, \tau_2, \dots, \tau_k \le 1.$$

An example where $k = 3$ is shown in Figure 2. As in the 2-dimensional example, one specifies m-dimensional faces of $\mathcal{A}$ by setting $k - m$ of the τ_i's to 0 or 1. We will consider a k-dimensional parallelepiped as *degenerate* precisely when $\{a_1, \dots, a_k\}$ is a linearly dependent set of vectors.

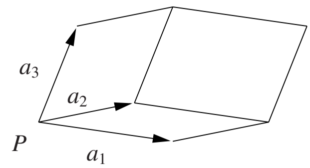


Figure 2. A 3-dimensional parallelepiped.

We will not in general distinguish between parallelepipeds with different base points so long as they have the same "edges." Because of this, we can write things like "the parallelepiped associated with the k-tuple of vectors $(a_1, \dots, a_k)$" or "the k-dimensional parallelepiped with matrix $A = (a_1, \dots, a_k)$."

Orientation. We want our parallelepipeds to be not just point sets but to have an orientation as well. We will define *orientation* for ordered k-tuples $(a_1, \dots, a_k)$ of vectors in $\mathbb{R}^n$, and this attaches an orientation to the parallelepiped associated with $(a_1, \dots, a_k)$.

Consider the directed line segments in Figure 3. We see that a and b have the same orientation, that c has the opposite orientation to a and b, and that, because it is not parallel to them, the orientation of d is not comparable to that of a, b, and c. If we translate the directed line segments to the origin and think of them as vectors, we can say that a, b, and c are comparable because they all lie in a common 1-dimensional vector subspace V of $\mathbb{R}^n$ while d is non-comparable because it is not in V. Also, a and b have the same orientation because $a \cdot b > 0$ while c has the opposite orientation because $a \cdot c < 0$ and $b \cdot c < 0$.

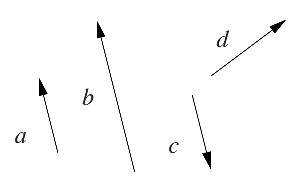


Figure 3. Comparable and non-comparable orientations.

We generalize this idea to ordered k-tuples of vectors:

Definition 1. Let $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ be two ordered k-tuples of vectors from $\mathbb{R}^n$.

1. If $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ are both linearly dependent sets, then $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ are considered to have the same orientation, the *0-orientation*.
2. Suppose $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ are both linearly independent sets and lie in the same k-dimensional vector subspace V of $\mathbb{R}^n$. Then $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ have the *same (nonzero) orientation* provided $\det(a_i \cdot b_j)_{k\times k} > 0$. If $\det(a_i \cdot b_j)_{k\times k} < 0$, then they have *opposite orientations*. (Note: It is easily seen that we cannot have $\det(a_i \cdot b_j) = 0$.)
3. In all other circumstances, the orientations of the two k-tuples are *non-comparable*.

Example 1. Let us take vectors

$$a_1 = (1, 0, -1), \quad a_2 = (1, -1, 0), \quad b_1 = (0, 1, -1), \quad b_2 = (1, 1, -2)$$

in $\mathbb{R}^3$. These all lie in the 2-dimensional vector subspace defined by the equation $x_1 + x_2 + x_3 = 0$, so we can compare the orientations of (a_1, a_2) and (b_1, b_2). Since $\det(a_i \cdot b_j) = 3$, we see that (a_1, a_2) and (b_1, b_2) have the same orientation.

We gain some intuition as to the meaning of "same orientation" if we imagine we are "looking down" on the plane $x_1 + x_2 + x_3 = 0$ (see Figure 4). Notice that a_1 rotates into a_2 and b_1 rotates into b_2 on their respective parallelograms in the *counterclockwise* direction. If the rotations had been in opposite directions, then the parallelograms would have had opposite orientations.

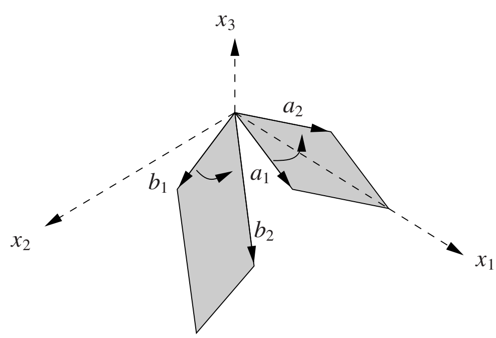


Figure 4. Looking "down" on (a_1, a_2) and (b_1, b_2).

Example 2. Let

$$a_1 = (1, 1, -2), \quad a_2 = (-3, -3, 6), \quad b_1 = (1, 0, -1), \quad b_2 = (0, 0, 0).$$

We see that (a_1, a_2) and (b_1, b_2) have the same orientation, the 0-orientation.

Example 3. If $\{a_1, \dots, a_k\}$ is a set of linearly independent vectors, then switching a_i and a_j for distinct i and j changes the orientation of $(a_1, \dots, a_k)$. Thus (a_1, a_2, a_3) and (a_2, a_1, a_3) have opposite orientations, but (a_1, a_2, a_3) and (a_3, a_1, a_2) have the same orientation.

Remark 1. Suppose $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ are linearly independent sets determining the same k-dimensional vector subspace V of $\mathbb{R}^n$. If F is the $k \times k$ matrix of a linear transformation of V to V carrying each a_i to b_i, then it can be shown that $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ have the same orientation if and only if $\det F > 0$.

From this point on, we shall feel free to refer to "the oriented parallelepiped $(a_1, \dots, a_k)$" where $a_1, \dots, a_k$ are vectors in $\mathbb{R}^n$.

Volume. Let $\mathcal{A}$ be the k-dimensional parallelepiped $(a_1, \dots, a_k)$.

Definition 2. We define the k-dimensional volume of $\mathcal{A}$ by $\mathrm{vol}(a_1, \dots, a_k) = \sqrt{\det(a_i \cdot a_j)}$.

We need to see that this definition makes sense calculationally and geometrically.

Suppose that $\{a_1, \ldots, a_k\}$ is a set of vectors in $\mathbb{R}^n$ and V is a k-dimensional vector subspace containing the a_i's. Let A be the $k \times k$ matrix $A = (a_1, \ldots, a_k)$ calculated with respect to an orthonormal basis of V. Then $\det(a_i \cdot a_j) = \det(A^T A) = (\det(A))^2 \geq 0$, so that $\sqrt{\det(a_i \cdot a_j)}$ can always be evaluated.

Notice that $\mathrm{vol}\,(a_1, \ldots, a_k) > 0$ if and only if $\{a_1, \ldots, a_k\}$ is a linearly independent set. Thus we also have the following:

Proposition 1. *The k-dimensional parallelepiped associated with the k-tuple of vectors $(a_1, \ldots, a_k)$ is nondegenerate (that is, $\{a_1, \ldots, a_k\}$ is a linearly independent set) if and only if vol $(a_1, \ldots, a_k) > 0$.*

Example 4. Let $a_1 = (1, 0, 0)$ and $a_2 = (0, 1, 1)$. These are orthogonal vectors in $\mathbb{R}^3$, so the area of the parallelogram associated with (a_1, a_2) must be $|a_1|\,|a_2| = \sqrt{2}$. This turns out to be precisely $\sqrt{\det(a_i \cdot a_j)}$.

Example 5. More generally, it is a standard exercise in a calculus text (see, for example, [**8**]) to show that the area of the parallelogram in $\mathbb{R}^3$ associated with (a_1, a_2) is given by $|a_1 \times a_2|$ where $\times$ is the standard vector product. It is straightforward to show that $|a_1 \times a_2|^2 = \det(a_i \cdot a_j)$.

Example 6. Again it is a standard exercise in a calculus text (see [**8**]) to see that the volume of the 3-dimensional parallelepiped in $\mathbb{R}^3$ associated with (a_1, a_2, a_3) is $|a_1 \cdot (a_2 \times a_3)| = |\det(A)|$ where A is the matrix $A = (a_1, a_2, a_3)$ expressed in terms of the basis vectors **i**, **j**, **k**. Then the volume is given by $|\det(A)| = \sqrt{\det(A^T A)} = \sqrt{\det(a_i \cdot a_j)}$.

Our definition of volume uses the *Gram determinant*. A good discussion of this topic and its relation to volume is found in [**2**] and [**7**]. There is a discussion of parallelepiped volume from a somewhat different viewpoint in [**4**].

The definition of simple *k*-vector.

Definition 3. By the *simple k-vector* $a_1 \wedge \cdots \wedge a_k$, where $a_1, \ldots, a_k \in \mathbb{R}^n$, we mean the set of all ordered k-tuples $(b_1, \ldots, b_k)$ such that $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ have the same orientation and volume. We indicate the relation between these two k-tuples by writing $(a_1, \ldots, a_k) \sim (b_1, \ldots, b_k)$.

If $(a_1, \ldots, a_k)$ has the 0-orientation (or, equivalently, $\mathrm{vol}(a_1, \ldots, a_k) = 0$), then we write $a_1 \wedge \cdots \wedge a_k = 0$.

Remark 2. $a_1 \wedge \cdots \wedge a_k = 0$ if and only if $\{a_1, \ldots, a_k\}$ is a linearly dependent set. As a consequence, if $a_1, \ldots, a_k \in \mathbb{R}^n$ and $k > n$, then $a_1 \wedge \cdots \wedge a_k = 0$. In particular, we have $a_1 \wedge \cdots \wedge a_k = 0$ whenever $a_i = a_j$ for some $i \neq j$.

On the other hand, if $a_1 \wedge \cdots \wedge a_k = b_1 \wedge \cdots \wedge b_k \neq 0$, then $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ must both span the same k-dimensional vector subspace of $\mathbb{R}^n$.

Definition 4. If V is a vector subspace of $\mathbb{R}^n$, we will say that a is a simple k-vector in V (or lies in V) if we can write $a = a_1 \wedge \cdots \wedge a_k$ where each $a_i \in V$.

We see from the last remark that every nontrivial simple k-vector lies in a uniquely determined k-dimensional subspace; however, trivial simple k-vectors lie in every vector subspace of $\mathbb{R}^n$.

Example 7. Consider various simple 2-vectors lying in a common plane as in Figure 5. The orientations are represented by curved arrows. We have $a_1 \wedge a_2 = (2a_1) \wedge (\frac{1}{2}a_2) = b_1 \wedge b_2$. We may think of $b_1 \wedge b_2$ as obtained from $a_1 \wedge a_2$ by a rotation in the plane; neither orientation nor area is changed. On the other hand, $a_1 \wedge a_2 \neq a_2 \wedge a_1$ or $(3a_1) \wedge a_2$—in the first case because of different orientations and in the second because of different areas.

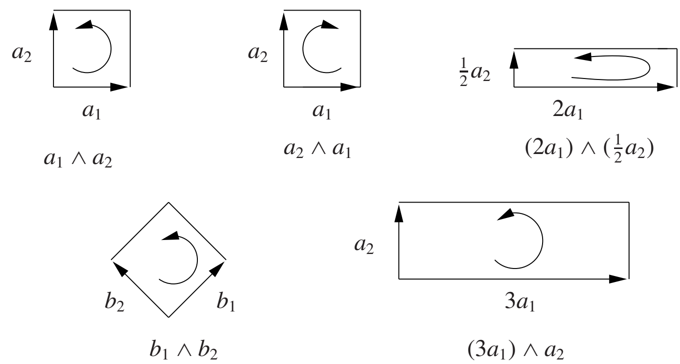


Figure 5. 2-vectors in the plane.

Example 8. In $\mathbb{R}^3$ we see that $\mathbf{i} \wedge \mathbf{j} \neq \mathbf{i} \wedge \mathbf{k}$ since the simple 2-vectors do not lie in a common 2-dimensional vector subspace.

The following lemma will be useful later.

Lemma 1. *Suppose that $a_1, \dots a_k, b_1, \dots b_k$ lie in V, a k-dimensional vector subspace of $\mathbb{R}^n$. Let A and B be $k \times k$ matrices $A = (a_1, \dots, a_k)$ and $B = (b_1, \dots, b_k)$ expressed with respect to an orthonormal basis of V, and suppose that F is a $k \times k$ matrix such that $B = FA$. Then we have the following:*

1. *If $a_1 \wedge \cdots \wedge a_k = b_1 \wedge \cdots \wedge b_k \neq 0$, then $\det A = \det B$ and $\det F = 1$.*
2. *If $\det F = 1$, then $a_1 \wedge \cdots \wedge a_k = b_1 \wedge \cdots \wedge b_k$.*

Proof. Since $B^T B = A^T F^T B = A^T F^T F A$, if we do some grouping and regrouping and take determinants, we obtain

$$\det(b_i \cdot b_j) = \det F\ \det(a_i \cdot b_j) = (\det F)^2\ \det(a_i \cdot a_j)$$

and

$$(\det B)^2 = \det F\ \det A\ \det B = (\det F)^2\,(\det A)^2.$$

If $a_1 \wedge \cdots \wedge a_k = b_1 \wedge \cdots \wedge b_k \neq 0$, then $\det(b_i \cdot b_j) = \det(a_i \cdot a_j) > 0$ and $\det(a_i \cdot b_j) > 0$. This forces $\det F = 1$ and $\det A = \det B \neq 0$.

Suppose that $\det F = 1$. Then $\{a_1, \dots, a_k\}$ is linearly dependent if and only if $\{b_1, \dots, b_k\}$ is, in which case $a_1 \wedge \cdots \wedge a_k = b_1 \wedge \cdots \wedge b_k = 0$. If $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ are independent, then we have $\det A,\ \det B \neq 0$. From this it follows that $\det(a_i \cdot a_j) = \det(b_i \cdot b_j) > 0$ and $\det(a_i \cdot b_j) > 0$, and we are done. ■

4. OPERATIONS ON SIMPLE k-VECTORS. Although we have introduced simple k-vectors, they are not yet true vectors in a vector space.

It turns out to be convenient, before embedding them in a vector space, to show that we can carry out some operations on simple k-vectors that we normally carry out on vectors—for example, multiplication by scalars or the dot product. We shall also show how to take a wedge product of simple k- and m-vectors to obtain simple $(k + m)$-vectors. The important step in these definitions is to show that they are well-defined, since we operate with representatives $(a_1, \ldots, a_k)$ of the simple k-vectors.

Once we have defined these operations, we shall show how to add simple k-vectors and obtain a vector space.

Scalar multiplication.

Definition 5. For $\lambda \in \mathbb{R}$ and a simple k-vector $a_1 \wedge \cdots \wedge a_k$, we define $\lambda(a_1 \wedge \cdots \wedge a_k) = a_1 \wedge \cdots \wedge \lambda a_i \wedge \cdots \wedge a_k$ for $i = 1, \ldots, k$. By $-(a_1 \wedge \cdots \wedge a_k)$ we shall mean $(-1)(a_1 \wedge \cdots \wedge a_k)$.

In Figure 6 we show two different ways we can represent $2(a_1 \wedge a_2)$ as an oriented parallelogram. Of course the two oriented parallelograms are equivalent.

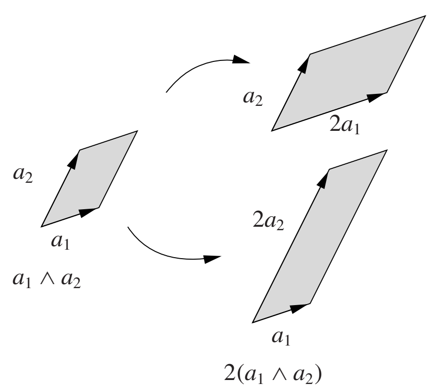


Figure 6. Two representations of a scalar product.

Proposition 2. *Multiplication of a simple k-vector by a scalar is well-defined.*

Proof. We suppose $\lambda \neq 0$ and $\{a_1, \ldots, a_k\}$ an independent set since otherwise everything is trivial.

Consider the case where $i = 1$ and assume $(a_1, \ldots, a_k) \sim (b_1, \ldots, b_k)$. Set $(c_1, \ldots, c_k) = (\lambda a_1, \ldots, a_k)$ and $(d_1, \ldots, d_k) = (\lambda b_1, \ldots, b_k)$. Clearly

$$\operatorname{span}\{c_1, \ldots, c_k\} = \operatorname{span}\{d_1, \ldots, d_k\}.$$

We easily compute

$$\operatorname{vol}(c_1, \ldots, c_k) = |\lambda|\sqrt{\det(a_i \cdot a_j)} = |\lambda|\sqrt{\det(b_i \cdot b_j)} = \operatorname{vol}(d_1, \ldots, d_k)$$

and $\det(c_i \cdot d_j) = \lambda^2 \det(a_i \cdot b_j) > 0$ so that $(c_1, \ldots, c_k) \sim (d_1, \ldots, d_k)$. Thus scalar multiplication makes sense if $i = 1$.

For general i and j, we set

$$(c_1, \dots, c_k) = (a_1, \dots, \lambda a_i, \dots, a_k) \quad \text{and} \quad (d_1, \dots, d_k) = (a_1, \dots, \lambda a_j, \dots, a_k)$$

and show as before that $(c_1, \dots, c_k) \sim (d_1, \dots, d_k)$. ∎

Remark 3. $0(a_1 \wedge \cdots \wedge a_k) = 0$.

Remark 4. Scalar multiplication has an obvious geometric interpretation: If you multiply one edge of an oriented parallelepiped by λ, then the volume changes by a factor of $|\lambda|$. If $\lambda > 0$, then the orientation is unchanged, but if $\lambda < 0$, then the orientation is reversed.

We now show that the set of simple k-vectors in a k-dimensional vector subspace can be thought of as a one-dimensional space.

Proposition 3. *If $a \neq 0$ is a simple k-vector in a k-dimensional vector subspace V of $\mathbb{R}^n$, then every simple k-vector in V is a scalar multiple of a.*

Specifically, if $a = a_1 \wedge \cdots \wedge a_k \neq 0$ and if $b = b_1 \wedge \cdots \wedge b_k$ lies in V, then $b = \alpha a$ where $\alpha = \det(\alpha_{ij})$ and $b_i = \sum_{j=1}^{k} \alpha_{ij}\, a_j$ for $i = 1, \dots, k$.

Proof. Since $\{a_1, \dots, a_k\}$ is a basis for V, the α_{ij}'s are uniquely determined, and we set $\alpha = \det(\alpha_{ij})$. We know that $\alpha a = (\alpha a_1) \wedge a_2 \wedge \cdots \wedge a_k$. Let us set $(c_1, \dots, c_k) = (\alpha a_1, a_2, \dots, a_k)$.

If $\alpha = 0$, then $\{b_1, \dots, b_k\}$ and $\{c_1, \dots, c_k\}$ are linearly dependent sets and $b = \alpha a = 0$.

Now assume $\alpha \neq 0$. It is easily shown that

$$\det(b_i \cdot b_j) = \alpha \det(a_i \cdot b_j) = \alpha^2 \det(a_i \cdot a_j).$$

This amounts to $\det(b_i \cdot b_j) = \det(c_i \cdot b_j) = \det(c_i \cdot c_j)$. Since $\alpha^2 \det(a_i \cdot a_j) > 0$, we see that $\det(c_i \cdot b_j) = \det(b_i \cdot b_j) > 0$. Thus $(b_1, \dots, b_k) \sim (c_1, \dots, c_k)$ and we are done. ∎

The dot product.

Definition 6. The *dot product* of two simple k-vectors is given by

$$(a_1 \wedge \cdots \wedge a_k) \cdot (b_1 \wedge \cdots \wedge b_k) = \det \begin{pmatrix} a_1 \cdot b_1 & \dots & a_1 \cdot b_k \\ \vdots & & \vdots \\ a_k \cdot b_1 & \dots & a_k \cdot b_k \end{pmatrix}.$$

Proposition 4. *The dot product of simple k-vectors is well-defined.*

Proof. It suffices to show that if $b_1 \wedge \cdots \wedge b_k = c_1 \wedge \cdots \wedge c_k \neq 0$, then $\det(a_i \cdot b_j) = \det(a_i \cdot c_j)$.

Let V be the k-dimensional vector subspace of $\mathbb{R}^n$ spanned by both $\{b_1, \dots, b_k\}$ and $\{c_1, \dots, c_k\}$. Let $\{v_1, \dots, v_k\}$ be an orthonormal basis for V and extend it to an orthonormal basis $\{v_1, \dots, v_n\}$ for $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ we can write $x = \sum_{i=1}^{n} \chi_i v_i$ and then define the *orthogonal projection* of x onto V by $x' = \sum_{i=1}^{k} \chi_i v_i$. Notice that if $y \in V$, then $x \cdot y = x' \cdot y$.

Let A', B, and C be the $k \times k$ matrices $A' = (a_1{}', \dots, a_k{}')$, $B = (b_1, \dots, b_k)$, and $C = (c_1, \dots, c_k)$ computed with respect to the orthonormal basis for V, where $a_i{}'$ is the orthogonal projection of a_i onto V. By Lemma 1 we know that $\det B = \det C$. Then

$$\begin{aligned}\det(a_i \cdot b_j) &= \det(a_i{}' \cdot b_j) = \det(A'^T B) = (\det A')(\det B)\\ &= (\det A')(\det C) = \det(a_i{}' \cdot c_j) = \det(a_i \cdot c_j).\end{aligned} \qquad \blacksquare$$

Remark 5. If a and b are simple k-vectors and $\lambda \in \mathbb{R}$, then $\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b)$ and $a \cdot b = b \cdot a$. We also introduce the symbol $|a|$ for $\sqrt{a \cdot a}$ and call this the *magnitude* of a. Notice that if $a = a_1 \wedge \cdots \wedge a_k$, then $|a| = \sqrt{\det(a_i \cdot a_j)} = \mathrm{vol}(a_1, \dots, a_k)$.

Example 9. Some of the simplest 2-vectors in $\mathbb{R}^3$ are $\mathbf{i} \wedge \mathbf{j}$, $\mathbf{i} \wedge \mathbf{k}$, and $\mathbf{j} \wedge \mathbf{k}$. When we compute their dot products,

$$\begin{aligned}(\mathbf{i} \wedge \mathbf{j}) \cdot (\mathbf{i} \wedge \mathbf{j}) = (\mathbf{i} \wedge \mathbf{k}) \cdot (\mathbf{i} \wedge \mathbf{k}) = (\mathbf{j} \wedge \mathbf{k}) \cdot (\mathbf{j} \wedge \mathbf{k}) = 1,\\ (\mathbf{i} \wedge \mathbf{j}) \cdot (\mathbf{i} \wedge \mathbf{k}) = (\mathbf{i} \wedge \mathbf{j}) \cdot (\mathbf{j} \wedge \mathbf{k}) = (\mathbf{i} \wedge \mathbf{k}) \cdot (\mathbf{j} \wedge \mathbf{k}) = 0,\end{aligned}$$

then we see that they behave like an “orthonormal” set.

Example 10. To gain some feeling for the geometric significance of dot product, we construct an example of two oriented rectangles (or 2-vectors) in $\mathbb{R}^3$ at an angle of θ with one another.

We start with $\mathbf{i} \wedge \mathbf{j}$, an oriented, magnitude-1 rectangle in the x_1x_2-plane. We rotate $\mathbf{j}$ “up” out of the x_1x_2-plane by an angle θ, producing the vector $c = (0, \cos\theta, \sin\theta)$. We next rotate $\mathbb{R}^3$ about the x_3-axis by an amount ϕ, and applying this rotation to the vectors $\mathbf{i}$ and c; we obtain, respectively,

$$a = (\cos\phi, \sin\phi, 0) \quad \text{and} \quad b = (-\sin\phi\, \cos\theta,\ \cos\phi\, \cos\theta,\ \sin\theta).$$

Then $a \wedge b$ may be considered an oriented, magnitude-1 rectangle making an angle of θ with the x_1x_2-plane and with $\mathbf{i} \wedge \mathbf{j}$ (see Figure 7).

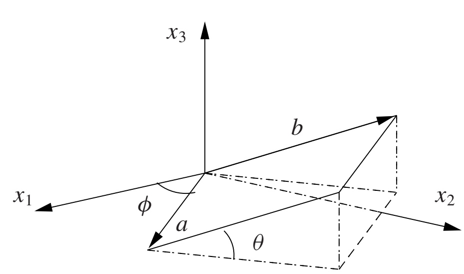


Figure 7. 2-vector at an angle θ with x_1x_2-plane.

We calculate their dot product:

$$(a \wedge b) \cdot (\mathbf{i} \wedge \mathbf{j}) = \cos\theta = |a \wedge b|\, |\mathbf{i} \wedge \mathbf{j}|\, \cos\theta.$$

Notice the resemblance to the dot product of ordinary vectors.

Remark 6. If one of the vectors of $\{a_1, \ldots, a_k\}$ is orthogonal to all of the vectors of $\{b_1, \ldots, b_k\}$, then $a_1 \wedge \cdots \wedge a_k$ must be "orthogonal" to $b_1 \wedge \cdots \wedge b_k$ in the sense that $(a_1 \wedge \cdots \wedge a_k) \cdot (b_1 \wedge \cdots \wedge b_k) = 0$.

We are familiar with the fact that in a vector space two vectors are equal if and only if all their components are equal and that in a Euclidean space components can be computed by taking dot products. A similar result holds for simple k-vectors.

Proposition 5. *If a and b are simple k-vectors, then $a = b$ if and only if $a \cdot c = b \cdot c$ for all simple k-vectors c.*

Proof. Suppose that $a \cdot c = b \cdot c$ for all simple k-vectors $c \in \mathbb{R}^n$. Then $a \cdot a = a \cdot b = b \cdot b$, which, if $a = a_1 \wedge \cdots \wedge a_k$ and $b = b_1 \wedge \cdots \wedge b_k$, amounts to $\det(a_i \cdot a_j) = \det(a_i \cdot b_j) = \det(b_i \cdot b_j)$. We see from this that $a \neq 0$ if and only if $b \neq 0$, and $\mathrm{vol}(a) = \mathrm{vol}(b)$. Further, if a and b lie in the same k-dimensional vector subspace, then they must have the same orientation.

The only possibility for $a \neq b$ is if $a, b \neq 0$ and $V = \mathrm{span}\{a_1, \ldots, a_k\} \neq W = \mathrm{span}\{b_1, \ldots, b_k\}$, so let us suppose that we have precisely that situation. We extend $\{a_1, \ldots, a_k\}$ to $\{a_1, \ldots, a_m\}$, a basis for $V + W$, in such a way that $a_{k+1}, \ldots, a_m$ are orthogonal to V. Next we construct an orthonormal basis $\{w_1, \ldots, w_k\}$ for W. We know by Proposition 3 that $b = (\beta w_1) \wedge w_2 \wedge \cdots \wedge w_k$ for a suitable scalar β and that $\beta \neq 0$ since $b \neq 0$. We may, without loss of generality, suppose that w_1 has a nonzero a_{k+1} component, that is, that $w_1 \cdot a_{k+1} \neq 0$. Let us set $c = a_{k+1} \wedge w_2 \wedge \cdots \wedge w_k$. It is an easy computation that $b \cdot c = \beta w_1 \cdot a_{k+1} \neq 0$. However $a \cdot c = 0$ since a_{k+1} is orthogonal to V. This contradiction gives us the desired conclusion. ■

Wedge products of simple *k*- and *m*-vectors.

Definition 7. We define the wedge product of the simple k-vector $a_1 \wedge \cdots \wedge a_k$ and the simple m-vector $b_1 \wedge \cdots \wedge b_m$ by

$$(a_1 \wedge \cdots \wedge a_k) \wedge (b_1 \wedge \cdots \wedge b_m) = a_1 \wedge \cdots \wedge a_k \wedge b_1 \wedge \cdots \wedge b_m.$$

We give a "picture" of this operation in Figure 8.

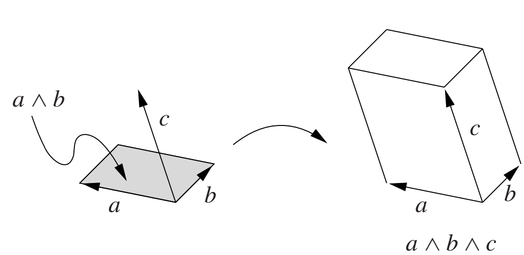


Figure 8. Wedge product of a 1- and a 2-vector.

Proposition 6. *The wedge product of a simple k- and m-vector is well-defined.*

Proof. It suffices to consider the case where $a_1 \wedge \cdots \wedge a_k = a_1' \wedge \cdots \wedge a_k'$ and show that $a_1 \wedge \cdots \wedge a_k \wedge b_1 \wedge \cdots \wedge b_m = a_1' \wedge \cdots \wedge a_k' \wedge b_1 \wedge \cdots \wedge b_m$.

If $\{a_1, \ldots, a_k, b_1, \ldots, b_m\}$ is a linearly dependent set, then the same must be true for $\{a'_1, \ldots, a'_k, b_1, \ldots, b_m\}$, and our desired equality follows trivially. So we may assume that $\{a_1, \ldots, a_k, b_1, \ldots, b_m\}$ and $\{a'_1, \ldots, a'_k, b_1, \ldots, b_m\}$ are independent sets.

Let V be the k-dimensional subspace $\operatorname{span}\{a_1, \ldots, a_k\} = \operatorname{span}\{a'_1, \ldots, a'_k\}$ and W the m-dimensional subspace $\operatorname{span}\{b_1, \ldots, b_m\}$. Next we construct an orthonormal basis $\{v_1, \ldots, v_{k+m}\}$ for $V \oplus W$ in such a way that $\{v_1, \ldots, v_k\}$ is a basis for V. (Because V and W are not necessarily orthogonal, it does not follow that $\{v_{k+1}, \ldots, v_{k+m}\}$ is a basis for W. See Figure 9.)

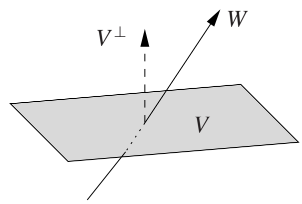


Figure 9. The subspaces V and W.

Let E and E' be the $(k+m) \times (k+m)$ matrices

$$E = (a_1, \ldots, a_k, b_1, \ldots, b_m) \quad \text{and} \quad E' = (a'_1, \ldots, a'_k, b_1, \ldots, b_m)$$

computed with respect to $\{v_1, \ldots, v_{k+m}\}$. Define $f : V \oplus W \to V \oplus W$ to be the linear transformation such that $f(a_i) = a'_i$ and $f(b_j) = b_j$ for all i, j, and take F to be the $(k+m) \times (k+m)$ matrix of f with respect to $\{v_1, \ldots, v_{k+m}\}$. We see that $E' = FE$. By Lemma 1, if we can show that $\det F = 1$, then we are done.

Let A and A' be the $k \times k$ matrices $A = (a_1, \ldots, a_k)$ and $A' = (a'_1, \ldots, a'_k)$ computed with respect to $\{v_1, \ldots, v_k\}$. If we consider the restriction of f to V, we have $f|_V : V \to V$. Let us take F' to be the $k \times k$ matrix of $f|_V$ computed with respect to $\{v_1, \ldots, v_k\}$. We see that $A' = F'A$ and that by Lemma 1, we have $\det F' = 1$.

Now let $\{w_1, \ldots, w_m\}$ be an orthonormal basis for W and set

$$\mathcal{B} = \{v_1, \ldots, v_k, w_1, \ldots, w_m\}.$$

Notice that $\mathcal{B}$ is a basis for $V \oplus W$ though not necessarily an orthonormal one. Suppose we compute the $(k+m) \times (k+m)$ matrix G of f with respect to $\mathcal{B}$. Since $f|_V$ carries V to V and $f|_W$ is the identity map, it is seen that

$$G = \begin{pmatrix} F' & 0 \\ 0 & I \end{pmatrix}$$

where I is the $m \times m$ identity matrix.

Since F and G are matrices of f with respect to different bases, we have $F = TGT^{-1}$ for some square matrix T. Therefore $\det F = \det G = \det F' = 1$, and we are done. ■

The following result is immediate and trivial:

Proposition 7. *The wedge product of simple k-, l-, and m-vectors is associative. That is,* $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

Remark 7. We originally introduced $a_1 \wedge \cdots \wedge a_k$ as the symbol for a simple k-vector. We can now also regard it as the product of k simple 1-vectors, parentheses being rendered unnecessary by the associative law.

Linear transformations acting on simple *k*-vectors. In this subsection and the next we briefly describe two topics for which, because we make no further use of them, we include no proofs. However they have such strong geometric content that we feel they merit our attention.

Proposition 8. *Suppose that $f : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation. If $a_1 \wedge \cdots \wedge a_k$ and $b_1 \wedge \cdots \wedge b_k$ are simple k-vectors in $\mathbb{R}^m$ such that $a_1 \wedge \cdots \wedge a_k = b_1 \wedge \cdots \wedge b_k$, then $f(a_1) \wedge \cdots \wedge f(a_k) = f(b_1) \wedge \cdots \wedge f(b_k)$.*

This result justifies the following:

Definition 8. If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then there is a transformation $\wedge^k f$ of simple k-vectors of $\mathbb{R}^m$ to simple k-vectors of $\mathbb{R}^n$ given by

$$\wedge^k f(a_1 \wedge \cdots \wedge a_k) = f(a_1) \wedge \cdots \wedge f(a_k).$$

Example 11. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $f(x_1, x_2) = (x_1 + 2x_2, x_2 - \frac{1}{2}x_1)$. Then the result of applying $\wedge^2 f$ to $\mathbf{i} \wedge \mathbf{j}$, namely, $\wedge^2 f(\mathbf{i} \wedge \mathbf{j}) = (\mathbf{i} - \frac{1}{2}\mathbf{j}) \wedge (2\mathbf{i} + \mathbf{j})$, is displayed in Figure 10.

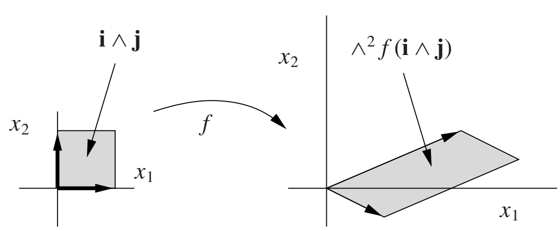


Figure 10. $\wedge^2 f$ applied to a 2-vector.

We cannot yet claim $\wedge^k f$ is itself a linear transformation, but it will turn out that we can treat it as such once we construct a vector space containing the simple k-vectors.

The Hodge star operator. Given a simple k-vector a in $\mathbb{R}^n$, we would like to construct a simple $(n-k)$-vector ${}^*a$ which is the "orthogonal complement" of a. It is much simpler to do this once one has constructed a vector space of k-vectors, so we are "cheating" by presenting it here, out of sequence.

Proofs of the following results may be found in [**1**] and [**4**]. They are valid in the vector spaces of k-vectors and may be stated without the adjective "simple," but we are not yet quite entitled to do that.

Proposition 9. *Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis for $\mathbb{R}^n$ and set $u = u_1 \wedge \cdots \wedge u_n$. For every simple k-vector a in $\mathbb{R}^n$, where $0 \le k \le n$, there is a unique simple $(n-k)$-vector ${}^*a$ which satisfies*

$$a \wedge b = ({}^*a \cdot b)\, u$$

for every simple $(n-k)$-vector b in $\mathbb{R}^n$.

The transformation $a \mapsto {}^*a$ is the *Hodge star operator*. Notice that we have to specify both $\mathbb{R}^n$ and u to define it. The choice of u is, in effect, a choice of "orientation" for $\mathbb{R}^n$. Figure 11 pictures the effect on simple 1- and 2-vectors in $\mathbb{R}^3$.

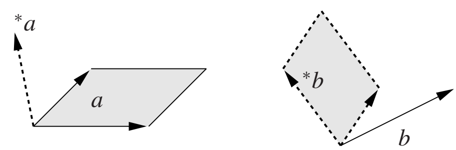


Figure 11. The star operator on 1- and 2-vectors in $\mathbb{R}^3$.

The following proposition exhibits the sense in which ${}^*a$ is an orthogonal complement to a. A proof can be found in [**4**].

Proposition 10. *Let u be as in Proposition 9. Choose a, a simple k-vector in $\mathbb{R}^n$. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for $\mathbb{R}^n$ with the property that $a = \alpha\,(v_1 \wedge \cdots \wedge v_k)$ for some scalar α, and let $\lambda = \pm 1$ such that $v_1 \wedge \cdots \wedge v_n = \lambda\, u$. (Such $v_1, \ldots, v_n$ and λ always exist.) Then*

$$ {}^*a = \lambda\,\alpha\,(v_{k+1} \wedge v_{k+2} \wedge \cdots \wedge v_n). $$

Example 12. Consider the orthonormal basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for $\mathbb{R}^3$ and, in the spirit of Proposition 9, take $u = \mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}$. Then it can be shown that the familiar *vector product* of vectors in $\mathbb{R}^3$ is given by $a \times b = {}^*(a \wedge b)$. In particular,

$$ \mathbf{i} \times \mathbf{j} = {}^*(\mathbf{i} \wedge \mathbf{j}) = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = {}^*(\mathbf{j} \wedge \mathbf{k}) = \mathbf{i}, \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = {}^*(\mathbf{k} \wedge \mathbf{i}) = \mathbf{j}. $$

5. THE VECTOR SPACE $\Lambda^k\mathbb{R}^n$. We do not yet know how to add simple k-vectors. It is the one operation we need to turn them into a vector space.

Suppose we take $\mathcal{S}_k$ to be the set of simple k-vectors in some fixed $\mathbb{R}^n$. For every simple k-vector a we define a map $\phi_a : \mathcal{S}_k \to \mathbb{R}$ by $\phi_a(b) = a \cdot b$. Notice there is no problem multiplying ϕ_a's by scalars or adding them: for $\lambda_i \in \mathbb{R}$ and simple k-vectors $a_i = a_{i1} \wedge \cdots \wedge a_{ik}$, we set

$$ (\lambda_1\phi_{a_1} + \cdots + \lambda_m\phi_{a_m})(b) = \lambda_1\, a_1 \cdot b + \cdots + \lambda_m\, a_m \cdot b $$

where $b \in \mathcal{S}_k$. For $1 \leq k \leq n$, if we take $\Lambda^k\mathbb{R}^n$ to be the set of finite linear combinations of ϕ_a's with the usual operations of function addition and multiplication by real numbers, then the following holds:

Theorem 1. $\Lambda^k\mathbb{R}^n$ *is a vector space over* $\mathbb{R}$.

Next notice that by Proposition 5 there is a one-to-one correspondence between $\{\phi_a : a \in \mathcal{S}_k\}$ and $\mathcal{S}_k$, so that we can, if we wish, identify ϕ_a and a. It is also easily seen that $\lambda\,\phi_a = \phi_{\lambda\, a}$, so that multiplication by a scalar amounts to the same thing whether dealing with simple k-vectors or elements of $\Lambda^k\mathbb{R}^n$. Because of this we will write

$$ \lambda_1\,\phi_{a_1} + \cdots + \lambda_m\,\phi_{a_m} \quad \text{as} \quad \lambda_1\, a_1 + \cdots + \lambda_m\, a_m $$

and we can, in a natural way, identify $\mathcal{S}_k$ with a subset of $\Lambda^k\mathbb{R}^n$. Thus we have solved the problem of adding simple k-vectors and forming them into a vector space.

(This tells us how to define $\Lambda^k\mathbb{R}^n$ for $1 \le k \le n$. We can add to this list by taking $\Lambda^k\mathbb{R}^n = \{0\}$ when $k > n$. The reason for this is that $a_1 \wedge \cdots \wedge a_k = 0$ if $k > n$. It is also standard and convenient to take $\Lambda^0\mathbb{R}^n$ to be $\mathbb{R}$.)

Next, for $\sum_i \lambda_i a_i \in \Lambda^k\mathbb{R}^n$ and b a simple k-vector, let us replace the function symbolism $\left(\sum_i \lambda_i a_i\right)(b)$ by $\left(\sum_i \lambda_i a_i\right) \cdot b$. Since $\left(\sum_i \lambda_i a_i\right) \cdot b = \sum_i \lambda_i (a_i \cdot b)$, we can extend the definition of dot product to $\Lambda^k\mathbb{R}^n$ by setting

$$\Big(\sum_i \lambda_i a_i\Big) \cdot \Big(\sum_j \xi_j b_j\Big) = \sum_{i,j} \lambda_i \xi_j \, (a_i \cdot b_j)$$

where the a_i's and b_j's are simple k-vectors. It must be checked that this is a well-defined extension. If we can write $b \in \Lambda^k\mathbb{R}^n$ both as $\sum_j \xi_j b_j$ and $\sum_j \xi'_j b'_j$, it is straightforward to see that $\sum_{i,j} \lambda_i \xi_j (a_i \cdot b_j) = \sum_{i,j} \lambda_i \xi'_j (a_i \cdot b'_j)$. We then have the following easy theorem:

Theorem 2. *If $\lambda \in \mathbb{R}$ and $a, b, c \in \Lambda^k\mathbb{R}^n$, where $1 \le k \le n$, then the following hold:*

1. $a \cdot b = b \cdot a$.
2. $\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b)$.
3. $a \cdot (b + c) = a \cdot b \, + \, a \cdot c$.

Example 13. Recall that in $\Lambda^2\mathbb{R}^3$ we have seen that $\{\mathbf{i} \wedge \mathbf{j},\, \mathbf{i} \wedge \mathbf{k},\, \mathbf{j} \wedge \mathbf{k}\}$ is an orthonormal set. Thus for $a = \alpha_1 \, (\mathbf{i} \wedge \mathbf{j}) + \alpha_2 \, (\mathbf{i} \wedge \mathbf{k}) + \alpha_3 \, (\mathbf{j} \wedge \mathbf{k})$ and $b = \beta_1 \, (\mathbf{i} \wedge \mathbf{j}) + \beta_2 \, (\mathbf{i} \wedge \mathbf{k}) + \beta_3 \, (\mathbf{j} \wedge \mathbf{k})$, by repeatedly applying Theorem 2, we have $a \cdot b = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$.

Next we want to find a basis for the vector space $\Lambda^k\mathbb{R}^n$.

Suppose we have simple k-vectors $a = a_1 \wedge \cdots \wedge a_k$ and $b = b_1 \wedge \cdots \wedge b_k$. Since $a \cdot b = \det(a_i \cdot b_j)$ and a is completely determined by values of the form $a \cdot b$, we see that a must be linear in each of the vectors $a_1, \ldots, a_k$. For example, for the first factor of a we have

$$(\lambda a_1) \wedge a_2 \wedge \cdots \wedge a_k = \lambda \, (a_1 \wedge a_2 \wedge \cdots \wedge a_k)$$

and

$$(a_1 + a'_1) \wedge a_2 \wedge \cdots \wedge a_k = (a_1 \wedge a_2 \wedge \cdots \wedge a_k) + (a'_1 \wedge a_2 \wedge \cdots \wedge a_k).$$

We can use this idea to find bases for $\Lambda^k\mathbb{R}^n$.

Example 14. Consider the special case $\Lambda^2\mathbb{R}^3$. Every simple 2-vector here has the form

$$a = (\alpha_1 \, \mathbf{i} + \alpha_2 \, \mathbf{j} + \alpha_3 \, \mathbf{k}) \wedge (\alpha'_1 \, \mathbf{i} + \alpha'_2 \, \mathbf{j} + \alpha'_3 \, \mathbf{k}).$$

Appealing repeatedly to linearity, we can rewrite this as

$$a = \alpha_1\alpha'_1(\mathbf{i} \wedge \mathbf{i}) + \alpha_1\alpha'_2(\mathbf{i} \wedge \mathbf{j}) + \alpha_1\alpha'_3(\mathbf{i} \wedge \mathbf{k}) + \cdots + \alpha_3\alpha'_3(\mathbf{k} \wedge \mathbf{k}).$$

Notice that 2-vectors with repetitions, such as $\mathbf{i} \wedge \mathbf{i}$, vanish, and other 2-vectors can be reversed if they are also sign-changed, as in $\mathbf{j} \wedge \mathbf{i} = -\mathbf{i} \wedge \mathbf{j}$. So we can ultimately write

$$a = \beta_{12}\,(\mathbf{i} \wedge \mathbf{j}) + \beta_{13}\,(\mathbf{i} \wedge \mathbf{k}) + \beta_{23}\,(\mathbf{j} \wedge \mathbf{k}) \tag{1}$$

for appropriate scalars β_{ij}. Since every 2-vector is a linear combination of simple ones, it follows that every element of $\Lambda^2\mathbb{R}^3$ has the form of (1). Moreover $\{\,\mathbf{i} \wedge \mathbf{j},\ \mathbf{i} \wedge \mathbf{k},\ \mathbf{j} \wedge \mathbf{k}\,\}$ is an independent set. To see this, suppose $c = \gamma_{12}\,(\mathbf{i} \wedge \mathbf{j}) + \gamma_{13}\,(\mathbf{i} \wedge \mathbf{k}) + \gamma_{23}\,(\mathbf{j} \wedge \mathbf{k}) = 0$. Then $\gamma_{12} = c \cdot (\mathbf{i} \wedge \mathbf{j}) = 0$, and, in a similar fashion, $\gamma_{13} = \gamma_{23} = 0$. Thus $\{\,\mathbf{i} \wedge \mathbf{j},\ \mathbf{i} \wedge \mathbf{k},\ \mathbf{j} \wedge \mathbf{k}\,\}$ is not only an orthonormal set but a basis for $\Lambda^2\mathbb{R}^3$.

The reasoning of Example 14 is readily extended and generalized to give us the following:

Theorem 3. *If $1 \le k \le n$ and $\{\,u_1, \dots, u_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then the set of k-vectors of the form $u_{i_1} \wedge \cdots \wedge u_{i_k}$, such that $i_1 < \cdots < i_k$, is an orthonormal basis for $\Lambda^k\mathbb{R}^n$.*

Example 15. If $\{\,u_1,\ u_2,\ u_3,\ u_4\}$ is an orthonormal basis for $\mathbb{R}^4$, then an orthonormal basis for $\Lambda^2\mathbb{R}^4$ consists of the elements

$$u_1 \wedge u_2, \quad u_1 \wedge u_3, \quad u_1 \wedge u_4, \quad u_2 \wedge u_3, \quad u_2 \wedge u_4, \quad u_3 \wedge u_4$$

while an orthonormal basis for $\Lambda^3\mathbb{R}^4$ consists of

$$u_1 \wedge u_2 \wedge u_3, \quad u_1 \wedge u_2 \wedge u_4, \quad u_1 \wedge u_3 \wedge u_4, \quad u_2 \wedge u_3 \wedge u_4.$$

Remark 8. Knowing Theorem 3 it is easily seen that if $\{\,u_1, \dots, u_n\}$ is any basis for $\mathbb{R}^n$, then $\{\,u_{i_1} \wedge \cdots \wedge u_{i_k} \ :\ i_1 < \cdots < i_k\}$ is a basis for $\Lambda^k\mathbb{R}^n$ and that $\dim \Lambda^k\mathbb{R}^n = \binom{n}{k}$.

We also want, of course, to extend the notion of wedge product to $\Lambda^k\mathbb{R}^n$. This can be done by setting

$$\Big(\sum_i \lambda_i a_i\Big) \wedge \Big(\sum_j \xi_j b_j\Big) = \sum_{i,j} \lambda_i \xi_j\,(a_i \wedge b_j)$$

where the a_i's and b_j's are simple k- and m-vectors respectively. As in the case of the dot product, it can be shown that the wedge product is well-defined; however the proof is not as simple and we do not go into it here. If we accept this definition, we can then easily establish the following:

Theorem 4. *Suppose that a and b are k- and m-vectors respectively, that c and d are r-vectors, and $\lambda \in \mathbb{R}$. Then the following hold:*

1. $\lambda\,(a \wedge b) = (\lambda\, a) \wedge b = a \wedge (\lambda\, b)$.
2. $a \wedge (c + d) = a \wedge c \,+\, a \wedge d$.
3. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.
4. $a \wedge b = (-1)^{km} b \wedge a$.

Example 16. It is no longer necessarily true in $\Lambda^k\mathbb{R}^n$ that all k-vectors are simple. If a is a simple k-vector, then it easily follows that $a \wedge a = 0$. But if, for example, we take an orthonormal basis $\{u_1, u_2, u_3, u_4\}$ for $\mathbb{R}^4$, then it is straightforward to show for $a = (u_1 \wedge u_2) + (u_3 \wedge u_4)$ that $a \wedge a \neq 0$.

6. APPLICATIONS.

k-vectors determined by $k+1$ points. We are familiar with vectors of the form A_0A_1 where A_0 and A_1 are points in $\mathbb{R}^n$. We generalize this to k-vectors of the form $A_0 \cdots A_k$, and it is convenient in that setting to picture k-vectors as oriented simplexes rather than oriented parallelepipeds.

The simplex S determined by $A_0, A_1, \dots, A_k$, points in $\mathbb{R}^n$, is

$$S = \left\{ t_0 A_0 + \cdots + t_k A_k \,:\, t_i \in \mathbb{R},\ 0 \le t_i,\ \sum_{i=0}^{k} t_i = 1 \right\}.$$

S is the smallest convex set containing $A_0, \dots, A_k$. We say that $A_0, \dots, A_k$ are the *vertices* of S. If we also endow S with an orientation, we can represent the class of oriented k-simplexes equivalent to it by a simple k-vector:

$$A_0A_1 \cdots A_k \stackrel{\text{def}}{=} \frac{1}{k!}(A_0A_1) \wedge (A_0A_2) \wedge \cdots \wedge (A_0A_k) \tag{2}$$

where each A_0A_i is a vector in $\mathbb{R}^n$ (see Figure 12). The factor $\frac{1}{k!}$ is the ratio of the volume of the simplex represented by $A_0A_1 \cdots A_k$ to that of the parallelepiped represented by $(A_0A_1) \wedge (A_0A_2) \wedge \cdots \wedge (A_0A_k)$.

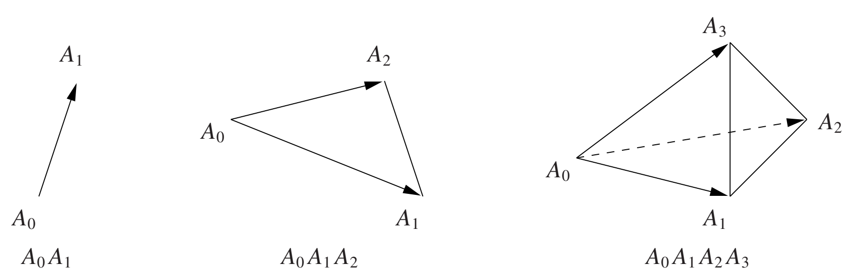


Figure 12. k-vectors of the form $A_0 \cdots A_k$.

Theorem 5. *For a simplex $A_0 \cdots A_k$ we have the following:*

1. *If we interchange A_i and A_j, where $i < j$, then*

$$A_0 \cdots A_j \cdots A_i \cdots A_k = -A_0 \cdots A_i \cdots A_j \cdots A_k.$$

2. *If $k \ge 2$, then*

$$\sum_{i=0}^{k} (-1)^i A_0A_1 \cdots \overline{A_i} \cdots A_k = 0,$$

where $\overline{A_i}$ denotes omission of A_i.

Proof. 1. First consider the case where $i = 0$ and $j = 1$. Notice that

$$A_0A_1 \cdots A_k = \frac{1}{k!}(A_0A_1) \wedge (A_0A_1 + A_1A_2) \wedge \cdots \wedge (A_0A_1 + A_1A_k)$$

$$= -\frac{1}{k!}(A_1A_0) \wedge (A_1A_2) \wedge \cdots \wedge (A_1A_k)$$
$$= -A_1A_0A_2 \cdots A_k.$$

The cases where $j > 1$ follow similarly. For all other values of i, where $j > i \geq 1$, the proof is an obvious consequence of (2).

2. Notice that

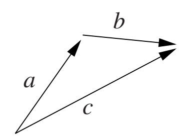
$$A_1A_2 \cdots A_k = \frac{1}{(k-1)!}(A_1A_2) \wedge (A_1A_3) \wedge \cdots \wedge (A_1A_k)$$
$$= \frac{1}{(k-1)!}(A_1A_0 + A_0A_2) \wedge (A_1A_0 + A_0A_3) \wedge \cdots \wedge (A_1A_0 + A_0A_k).$$

When we multiply this out, because $(A_1A_0) \wedge (A_1A_0) = 0$, we must have

$$A_1A_2 \cdots A_k = \frac{1}{(k-1)!}\Big\{(A_0A_2) \wedge (A_0A_3) \wedge \cdots \wedge (A_0A_k)$$
$$+ \sum_{i=2}^{k}(A_0A_2) \wedge \cdots \wedge (A_0A_{i-1}) \wedge (A_1A_0) \wedge (A_0A_{i+1}) \wedge \cdots \wedge (A_0A_k)\Big\}$$
$$= A_0A_2A_3 \cdots A_k$$
$$+ \frac{1}{(k-1)!}\left\{\sum_{i=2}^{k}(-1)^{i-1}(A_0A_1) \wedge (A_0A_2) \wedge \cdots \wedge \overline{(A_0A_i)} \wedge \cdots \wedge (A_0A_k)\right\}$$
$$= \sum_{i=1}^{k}(-1)^{i-1}A_0A_1 \cdots \overline{A_i} \cdots A_k. \qquad \blacksquare$$

This theorem and its proof can also be found in [**4**].

The law of vector addition is usually understood in terms of a triangle as in Figure 13. Part 2 of Theorem 5 can be interpreted as a generalization of the law of vector addition. It says, in effect, that if the $(k-1)$-dimensional faces of a k-simplex are thought of as $(k-1)$-vectors with the proper orientations, then the sum of these "face-vectors" is 0.



$c = a + b$

Figure 13. Law of vector addition.

Example 17. From Theorem 5, the "law of vector addition" for 2-simplexes is

$$A_1A_2A_3 - A_0A_2A_3 + A_0A_1A_3 - A_0A_1A_2 = 0.$$

We provide a picture of how this works in Figure 14 by "unfolding" a 3-simplex $A_0A_1A_2A_3$ into the plane and indicating with circular arrows on the "unfolded" version the orientation of each 2-dimensional face of the 3-simplex.

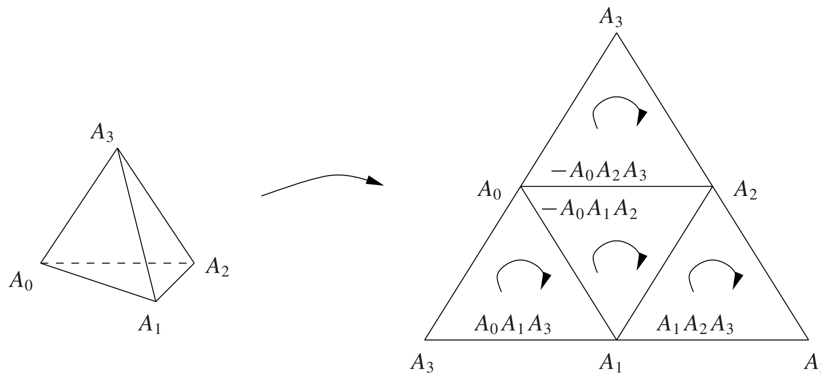


Figure 14. Vector addition law for 2-vectors.

The law of cosines. If $1 \le k \le n$, we know that $\Lambda^k\mathbb{R}^n$ is isomorphic to $\mathbb{R}^m$ where $m = \binom{n}{k}$. It is then easy to see that the Schwarz inequality holds in $\Lambda^k\mathbb{R}^n$, that is, that $|a \cdot b| \le |a|\,|b|$ if a and b are k-vectors in $\mathbb{R}^n$. Thus we can do the following:

Definition 9. If a, b are nonzero k-vectors in $\mathbb{R}^n$, then we define the *angle between a and b* to be the unique θ such that $0 \le \theta \le \pi$ and

$$\cos\theta = \frac{a \cdot b}{|a|\,|b|}.$$

Example 18. If we glance back at Example 10 where the 2-vectors $\mathbf{i} \wedge \mathbf{j}$ and $a \wedge b$ were constructed to have the angle θ between them in the same sense as two planes in $\mathbb{R}^3$, we see that we also have an angle of θ between them in the sense of Definition 9.

For an n-simplex $A_0 \dots A_n$, $n \ge 2$, it is then possible to talk about the angle between any two faces and to generalize the law of cosines from triangles to simplexes. (This result is stated in [**3**].) Let F_i be the $(n-1)$-dimensional face $A_0 \dots \overline{A_i} \dots A_n$ that does not contain the vertex A_i, and let θ_{ij} be the angle between F_i and F_j.

Theorem 6 (Law of cosines for n-simplexes). *For any face F_i of a simplex $A_0 \dots A_n$ we have*

$$|F_i|^2 = \sum_{j \ne i} |F_j|^2 + 2\sum_{\substack{j<l \\ j,l \ne i}} (-1)^{j+l} |F_j||F_l| \cos\theta_{jl}.$$

Proof. Since $|F_i|^2 = F_i \cdot F_i$, by the second part of Theorem 5 we may write

$$\begin{aligned} |F_i|^2 &= \left(\sum_{j \ne i} (-1)^{j+1} F_j\right) \cdot \left(\sum_{l \ne i} (-1)^{l+1} F_l\right) \\ &= \sum_{j \ne i} |F_j|^2 + 2\sum_{\substack{j<l \\ j,l \ne i}} (-1)^{j+l} |F_j||F_l| \cos\theta_{jl}, \end{aligned}$$

where $0 \le j, l \le n$. ∎

The parallelogram law. The parallelogram law states that for any given parallelogram,

$$\sum \text{sides}^2 = \sum \text{diagonals}^2.$$

We generalize this to n-dimensional parallelepipeds.

Suppose we have an n-dimensional parallelepiped $\mathcal{P}$ with the vectors $a_1, \ldots, a_n$ as its edges. If the origin is a vertex of the parallelepiped, then, in general, by *vertices* of $\mathcal{P}$ we mean points of the form

$$A = \sum_{i=1}^{n} \beta_i a_i \quad \text{where } \beta_i = 0, 1,$$

and we will call $(\beta_1, \ldots, \beta_n)$ the *binary sequence associated with* A. We say that two vertices A and B are *adjacent* provided $AB = \pm a_i$ for some i.

We define $F_i = a_1 \wedge \cdots \wedge \overline{a_i} \wedge \cdots \wedge a_n$ and we want to consider this to be a face of $\mathcal{P}$. However we must be careful to notice here that we are treating a face as an $(n-1)$-vector rather than a point-set, and we could conceivably have two different parallel $(n-1)$-dimensional faces, $\mathcal{F}$ and $\mathcal{F}'$, considered as point-sets, as in Figure 15, which are associated with the same face F_i considered as an $(n-1)$-vector. This is an important distinction; in the formula we gave for the 2-dimensional version of the parallelogram law, the faces of the paralellogram were treated as line segments, that is, as point-sets. In the generalization we shall construct, it is convenient to treat faces as simple $(n-1)$-vectors, and this will lead to a somewhat different formula.

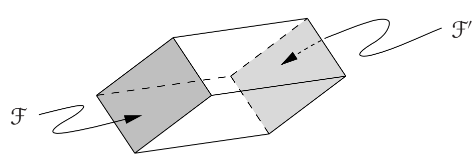


Figure 15. Different point-set faces.

In this n-dimensional setting, we consider diagonals of $\mathcal{P}$ to be oriented simplexes associated with vertices. If A is a vertex of $\mathcal{P}$, then we define the associated *diagonal* to be the simple $(n-1)$-vector

$$D_A = A_1 \cdots A_n \quad \text{where } AA_i = \pm a_i \text{ for } i = 1, \ldots, n.$$

Thus $A_1, \ldots, A_n$ are the vertices adjacent to A and the requirement that $AA_i = \pm a_i$ specifies each one uniquely. See Figure 16.

Let us note some simple relations between the objects we have introduced: Suppose A has the associated binary sequence $(\beta_1, \ldots, \beta_n)$. If $\beta_i = 0$, it is not hard to see that $AA_i = a_i$, while if $\beta_i = 1$, we have $AA_i = -a_i$. Thus

$$AA_i = (-1)^{\beta_i} a_i.$$

Set $\iota_A = (-1)^{\beta_1 + \cdots + \beta_n}$. It follows from (2) that

$$F_i = a_1 \wedge \cdots \wedge \overline{a_i} \wedge \cdots \wedge a_n = (-1)^{\beta_i} \, \iota_A \, (n-1)! \, AA_1 \cdots \overline{A_i} \cdots A_n. \tag{3}$$

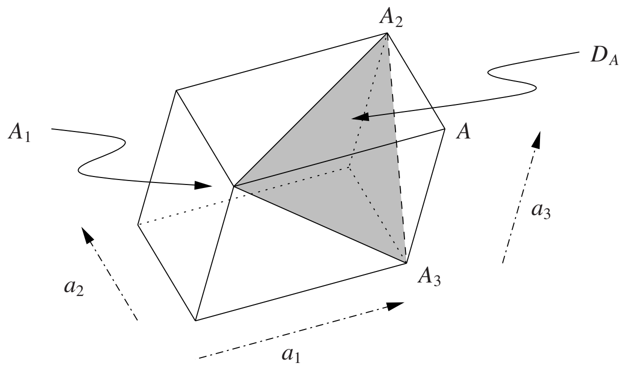


Figure 16. A diagonal of $\mathcal{P}$.

By Theorem 5, we have $A_1A_2\cdots A_n = \sum_{i=1}^{n}(-1)^{i+1}AA_1\cdots\overline{A_i}\cdots A_n$. From the definition of D_A and (3), we obtain

$$D_A = \sum_{i=1}^{n}\frac{(-1)^{\beta_i+i+1}\iota_A}{(n-1)!}\,F_i. \tag{4}$$

Thus, for example, in Figure 16, we have $n=3$, $(\beta_1,\beta_2,\beta_3)=(1,0,1)$, $\iota_A=1$, and $D_A = -\frac{1}{2}\,(F_1+F_2+F_3)$.

Theorem 7 (Parallelogram law). *For an n-dimensional parallelepiped $\mathcal{P}$, $n\geq 2$, with faces F_i, vertices A, and diagonals D_A, we have*

$$\sum_A |D_A|^2 = \frac{2^n}{((n-1)!)^2}\sum_{i=1}^{n}|F_i|^2. \tag{5}$$

Proof. Let us suppose that for each vertex A the associated binary sequence is $(\beta_1^A,\ldots,\beta_n^A)$. Appealing to (4), we have

$$\sum_A |D_A|^2 = \sum_A D_A\cdot D_A = \frac{1}{((n-1)!)^2}\sum_{i,j=1}^{n}(-1)^{i+j}\sum_A(-1)^{\beta_i^A+\beta_j^A}\,(F_i\cdot F_j). \tag{6}$$

Fix i,j such that $i\neq j$. Notice that $(\beta_i^A,\beta_j^A) = (0,0)$, $(0,1)$, $(1,0)$, or $(1,1)$ and each possibility occurs in exactly one quarter of the terms of (6) which contain i and j. Thus

$$\sum_A(-1)^{\beta_i^A+\beta_j^A}\,(F_i\cdot F_j) = 0 \quad \text{whenever} \quad i\neq j.$$

Since the total number of vertices is 2^n, we see that (6) reduces to (5). ∎

The Pythagorean theorem. In its higher-dimensional version, the Pythagorean theorem can be thought of as saying the following: Given an orthogonal n-simplex, $n\geq 2$, the volume squared of its “oblique” face is the sum of the volumes squared of the other faces. We say that an n-simplex $A_0A_1\cdots A_n$ is *orthogonal* if, for some labelling of its vertices, the angle between the faces $A_0\cdots\overline{A_j}\cdots A_n$ and $A_0\cdots\overline{A_l}\cdots A_n$ is $\pi/2$ whenever $1\leq j,l\leq n$ and $j\neq l$. (Here we mean angle in the sense of Definition 9.) Under these circumstances, we call $A_1\cdots A_n$ the *oblique face* of the simplex.

Example 19. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis for $\mathbb{R}^n$. If A_0 is the origin and $A_i = u_i$ for $1 \leq i \leq n$, then $A_0 \cdots \overline{A_i} \cdots A_n = \frac{1}{(n-1)!} u_1 \wedge \cdots \wedge \overline{u_i} \wedge \cdots \wedge u_n$. It is clear that $A_0 A_1 \cdots A_n$ is an orthogonal n-simplex.

Theorem 8. *If $A_0 A_1 \cdots A_n$, $n \geq 2$, is an orthogonal n-simplex with oblique face $A_1 \cdots A_n$, then*

$$|A_1 \cdots A_n|^2 = \sum_{i=1}^{n} |A_0 \cdots \overline{A_i} \cdots A_n|^2.$$

Proof. This follows from the law of cosines (Theorem 6) since $\cos(\theta_{jl})$ is 0 if $j \neq l$. ■

The Binet-Cauchy formula. As a final application we present a proof of a purely algebraic result, the Binet-Cauchy formula. See [**2**] or [**4**] for an alternative proof.

Suppose that $1 \leq k \leq n$ and we are given $k \times n$ and $n \times k$ matrices:

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \ldots & a_{kn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & \ldots & b_{1k} \\ \vdots & & \vdots \\ b_{n1} & \ldots & a_{nk} \end{pmatrix}.$$

We describe the $k \times k$ submatrices of A and B in the following way:

$$A_{i_1 \ldots i_k} = \begin{pmatrix} a_{1i_1} & \ldots & a_{1i_k} \\ \vdots & & \vdots \\ a_{ki_1} & \ldots & a_{ki_k} \end{pmatrix} \quad \text{and} \quad B_{i_1 \ldots i_k} = \begin{pmatrix} b_{i_1 1} & \ldots & b_{i_1 k} \\ \vdots & & \vdots \\ b_{i_k 1} & \ldots & a_{i_k k} \end{pmatrix}$$

where it is understood that $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$ and $i_1 < i_2 < \cdots < i_k$.

Theorem 9 (Binet-Cauchy).

$$\det(AB) = \sum_{i_1 < \cdots < i_k} \left(\det A_{i_1 \ldots i_k}\right) \left(\det B_{i_1 \ldots i_k}\right).$$

Proof. Let $\{u_1, \ldots, u_n\}$ be the standard orthonormal basis for $\mathbb{R}^n$, that is, $u_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is in the ith position. Set

$$a_i = \sum_{j=1}^{n} a_{ij} u_j \quad \text{and} \quad b_i = \sum_{j=1}^{n} b_{ji} u_j,$$

that is, the a_i's and b_i's are the row and column vectors of A and B respectively. Next take a and b to be the simple k-vectors

$$a = a_1 \wedge \cdots \wedge a_k \quad \text{and} \quad b = b_1 \wedge \cdots \wedge b_k.$$

We know by Theorem 3 that $\{u_{i_1} \wedge \cdots \wedge u_{i_k} : i_1 < \cdots < i_k\}$ is an orthonormal basis for $\Lambda^k \mathbb{R}^n$, so we should be able to write a and b in terms of this basis. Using the definition of dot product, we easily calculate that $a \cdot (u_{i_1} \wedge \cdots \wedge u_{i_k}) = \det A_{i_1 \ldots i_k}$ and that a similar result holds for b. Thus

$$a = \sum_{i_1 < \cdots < i_k} (\det A_{i_1 \ldots i_k})\, u_{i_1} \wedge \cdots \wedge u_{i_k} \quad \text{and} \quad b = \sum_{i_1 < \cdots < i_k} (\det B_{i_1 \ldots i_k})\, u_{i_1} \wedge \cdots \wedge u_{i_k}.$$

Finally we have

$$\det(AB) = \det(a_i \cdot b_j) = a \cdot b = \sum_{i_1 < \cdots < i_k} \left(\det A_{i_1 \ldots i_k}\right) \left(\det B_{i_1 \ldots i_k}\right),$$

and we are done.

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